INTRO TO ALGEBRAIC GEOMETRY, PROBLEM SET 4

Due Thursday October 14 in class (no lates). Hand in all the problems.

You're strongly encouraged to collaborate (although write up solutions separately), and you're also strongly encouraged to ask me questions (if you're stuck, or if the question is vaguely worded, or if you want to try out an argument).

Problems 2 and 3 are short once you realize what's going on. Problem 4 is the easiest. Problems 5 and 6 are very rewarding; don't tackle 7 before 5 and 6.

General questions.

1. Suppose $X \subset \mathbb{A}^n$ is an (affine) algebraic set, and $S \subset X$ is a subset. Show that if $\overline{S}$ is the closure of S in the Zariski topology (i.e. the smallest closed set containing S), then $\overline{S} = V(I(S))$.
2. In Class 7, we discussed the fact that if X is a variety, then $\mathcal{O}_X(U)$ is a subring of $k(X)$. One of the consequences I mentioned was that $\mathcal{O}_X(U \cap V)$ is the subring of $k(X)$ generated by $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ (if U and V are open subsets of X). This is wrong. Zuoliang gave a counterexample where X is the line with the doubled origin. Find a counterexample. (Notice how pathological this counterexample is. When we define *variety* to remove such pathologies ("non-separated" prevarieties), it is possible that $\mathcal{O}_X(U \cap V)$ is indeed the ring generated by $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$, and in fact it may be true that every non-separated prevariety provides a counterexample, and every counterexample is a non-separated prevariety.)
3. A prevariety X over $\overline{k}$ is *rational* if its function field $k(X)$ is isomorphic to $\overline{k}(t_1, \dots, t_n)$ for some n. Prove that if X is the hypersurface $wx = yz$ in $\mathbb{A}^4$, then X is rational.

Variations on the conic $x^2 + y^2 = z^2$.

For the next four questions, suppose $\operatorname{char} \overline{k} \neq 2$, and let C be the curve in $\mathbb{P}^2$ given by $x^2 + y^2 = z^2$, as discussed in class.

4. Let ρ be the projection $C \to \mathbb{P}^1$ given by $(x; y; z) \mapsto (x; y)$. If p is a point of $\mathbb{P}^1$, how many points does $\rho^{-1}p$ have? (There are two cases.)
5. (a) Show that $\mathbb{P}^2 \setminus (0; 1; 1) \to \mathbb{P}^1$ given by $(x; y; z) \mapsto (x; y - z)$ is a morphism. (This is projection from (0;1;1); do you see why?)
   (b) Hence $\pi' : C \setminus (0; 1; 1) \to \mathbb{P}^1$ given by $(x; y; z) \mapsto (x; y - z)$ is also a morphism (just compose the immersion of $C \setminus (0; 1; 1) \to \mathbb{P}^2 \setminus (0; 1; 1)$ with the projection of (a)). Describe a morphism $\pi : C \to \mathbb{P}^1$ which extends

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this morphism. (In other words, describe a morphism π whose restriction to $C \setminus (0;1;1)$ is π'.)
(c) Show that this is an isomorphism by giving a morphism $\sigma : \mathbb{P}^1 \to C$, and showing that $\sigma \circ \pi$ and $\pi \circ \sigma$ are the identity (on C and $\mathbb{P}^1$ respectively). (To show that the compositions are the identity, you need only show it pointwise, i.e. that for any point $p \in \mathbb{P}^1$, $\sigma(\pi(p)) = p$, and for any point $q \in C$, $\pi(\sigma(q)) = q$. Do you see why? So you need only think about sets, and not about topological spaces, or — worse still — structure sheaves.)

6. Using the isomorphism of the previous exercise, find all solutions to $x^2 + y^2 = z^2$ where x, y, and z lie in a field k of characteristic not 2 (not necessarily algebraically closed, e.g. $\mathbb{Q}$). (No justification is required, as we haven't really discussed algebraic objects over fields that are not algebraically closed. But explain where your formulas came from. Warning about a subtlety: would your formula give $(6, 8, 10)$ if $k = \mathbb{Q}$?)
7. (a) Let D be the prevariety $xy = x^3 + y^3$ in $\mathbb{A}^2$. Show that $D \setminus \{(0,0)\}$ is an affine variety.
(b) Find all rational points (i.e. points (x, y) where x and y are rational) on D. (No justification is required.)