INTRODUCTION TO ALGEBRAIC GEOMETRY, CLASS 10

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Contents

Problem sets can be picked up at my office; I'll also bring them in on Thursday.

This class is an aside!

1. Schemes

Given what we know about prevarieties, we can define schemes, and morphisms between them. Prevarieties can be interpreted as a special case (a very important one).

Tune out if you want.

If you tune in, here's what you should do. For the rest of the semester, try to follow everything I say about prevarieties and varieties in the language of schemes. In general, the proofs will "essentially" be the same. But watch out when I invoke the Nullstellensatz, which doesn't really generalize to schemes.

As references, I'd suggest (i) Mumford Ch. 2, (ii) Hartshorne II, and/or (iii) Eisenbud-Harris. Mumford Ch. 2 is the closest reference; as a caution, when his "prescheme" is my "scheme", and his "scheme" is my "separated scheme" (not that I'll discuss separatedness today); terminology has settled on the latter. (If you ever read Grothendieck, you have to watch out for this as well. If you follow what I say in the rest of the semester, but transpose everything to schemes, you'll essentially do Ch. 2.

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If you're thinking of roaring ahead, especially if you've seen some of this stuff before, or if you're thinking of going deeper into the field, you might prefer Hartshorne, which is much denser and more complete.

And Eisenbud-Harris is at least worth browsing; I'm a little less familiar with the beginning, so I can't specifically recommend particular parts, but the perspective is intuitive and geometric, and should link our perspective with Hartshorne's emphasis on theory.

Like prevarieties, schemes have points, topologies, and structure sheaves. Like prevarieties, we start by defining affine schemes.

I don't think you'll find the definitions extremely hard.

1.1. **Affine schemes.** Given a ring R. (Special case: a nilpotent-free finitely generated $\overline{k}$-algebras which are integral domains, which will essentially give us affine varieties. Nilpotent-free finitely generated $\overline{k}$-algebras will essentially give us (affine) algebraic sets.)

Examples: $\overline{k}[x, y]$, $\overline{k}[x]/x^2$, $\mathbb{Z}$. Prevarieties turn out to be affine schemes. More precisely, to every prevariety over $\overline{k}$ one can associate a scheme in a natural way. Most precisely, there is a functor from the category of prevarieties over $\overline{k}$ to the category of schemes (explain), which expresses the first category as a full subcategory of the second.

We define the topological space, denoted $\operatorname{Spec} R$.

Points are the prime ideals of R, denote $\operatorname{Spec} R$.

Exercise. Describe the points of $\operatorname{Spec} \mathbb{R}[x]$ (and compare to the points of $\mathbb{C}$ and $\mathbb{R}$).

In 3 examples: $\overline{k}[x, y]$, have old-fashioned points, corresponding to maximal ideals (Nullstellensatz). Have prime ideals like $(y - 3x^2)$, which correspond intuitively (and later, more precisely) to irreducible curves; this prime ideal is the *generic point* (move later). And have the prime ideal (0).

Each point $\mathfrak{p}$ has a *residue field*: $R_{\mathfrak{p}}$ is a local ring, with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ and residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. **This is perhaps better described as the fraction field of the domain $R/\mathfrak{p}$; I didn't say this in class.**

We now define the topology on $\operatorname{Spec} R$. The closed subsets are declared to be: given an ideal I, primes containing the ideal; we can call this $V(I)$, as we did before with prevarieties, once we've described the value of elements of R at points of $\operatorname{Spec} R$.

Example. If $\mathfrak{p}$ is the point of $\operatorname{Spec} k[x, y]$ corresponding to the prime ideal $(y - 3x^2)$, show that $\{\mathfrak{p}\}$ is not a closed subset in the Zariski topology, i.e. $\mathfrak{p}$ is not a *closed point.* Hint: $\mathfrak{p}$ is not a maximal ideal.

Can check that the closure of a point $\mathfrak{p}$ is an irreducible subset X; those prime ideals containing the ideal $\mathfrak{p}$. (Explain in case of $\overline{k}[x, y]$ and $\mathbb{Z}$.) Conversely, each irreducible closed subset corresponds to an ideal that you can check is prime. Hence points correspond to irreducible subsets, and the point corresponding to X is called the *generic point* of X.

Regular functions on an affine scheme.

Before getting down to brass tacks and defining the structure sheaf, it will be easier to get explicit and talk about regular functions, because these are a little weird.

In the case of prevarieties over $\overline{k}$, a regular function was a function to $\overline{k}$ with nice properties (locally quotient of polynomials). But with Spec, there is no particular $\overline{k}$. Functions take values in different fields at different points.

Examples: The function 24 on $\operatorname{Spec} \mathbb{Z}$. A function on Spec of the dual numbers. The functions on $\operatorname{Spec} \overline{k}[x, y]$.

The structure sheaf $\mathcal{O}_X$.

Theorem. There exists a sheaf $\mathcal{O}_X$ such that $\mathcal{O}_X(D(f)) = R_f$.

Proof omitted.

You can recover a sheaf from its sections on a base, so this lets you recover the entire sheaf. Proof is in Hartshorne II.2, and in Mumford II.

Stalk at a point if $R_{\mathfrak{p}}$.

Definition. An affine scheme is the data of a topological space X with a structure sheaf $\mathcal{O}_X$, where X is homeomorphic to $\operatorname{Spec} R$ for some R, and (via this homeomorphism) $\mathcal{O}_{\operatorname{Spec} R} \cong \mathcal{O}_X$.

Example. The rational functions $(x-2)/(x-1)^2$ on $\operatorname{Spec} \overline{k}[x]$; and $9/4$ on $\operatorname{Spec} \mathbb{Z}$. If an affine scheme is irreducible, then one can define its *function field* in the same way as we did for prevarieties: $k(\operatorname{Spec} R)$ is the quotient field of R, or the same construction with any affine open. Example: $\operatorname{Spec} \mathbb{Z}$, get $\mathbb{Q}$. Note that this is the residue field at the generic point; you can check that the function field is this in general. Discuss both examples.

1.2. Schemes.

A *scheme* is the data $(X, \mathcal{O}_X)$, where X is a topological space, and $\mathcal{O}_X$ is a sheaf, such that X can be expressed as a union of U_α, where $(U_\alpha, \mathcal{O}_{U_\alpha})$ is an affine scheme.

(Note: we no longer require *connectedness*, or *finiteness of the cover.*)

1.3. **Morphisms of affine schemes.** Remember that affine varieties over $\overline{k}$ corresponded to rings over $\overline{k}$ with certain nice properties, and that morphisms between them corresponded to ring maps (over $\overline{k}$), except the arrows were reversed.

In the same way, we'll define morphisms of affine schemes $\operatorname{Spec} S \to \operatorname{Spec} R$ to correspond to morphisms $f^* : R \to S$.

Explain where do points go: $\mathfrak{p}$ in S goes to $f^*\mathfrak{p}$. Check that it is prime. Do example of projection corresponding to $\overline{k}[t] \to \overline{k}[x, y]$, $t \mapsto x$.

You can check that this is a continuous map: if I is an ideal, then $f^* I$ is also an ideal.

Exercise. Suppose $\mathfrak{p}$ is a prime ideal of some ring R. Then the ring morphism $R \to R_{\mathfrak{p}}$ corresponds to a map of schemes $\pi : \operatorname{Spec} R_{\mathfrak{p}} \to \operatorname{Spec} R$. Show that π is injective. Thus the points of $R_{\mathfrak{p}}$ form a subset of the points of R; which prime ideals of R do they correspond to? (Feel free to quote results from commutative algebra if you want.)

Exercise. The ring morphism $\mathbb{Z} \to \mathbb{Z}[i]$ corresponds to a map of schemes $f : \operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}$. Suppose (p) is a prime ideal of $\mathbb{Z}$ (warning: p could be 0). Find the points of $f^{-1}(p)$ in $\operatorname{Spec} \mathbb{Z}[i]$. Compare the degree of the residue field extensions with the number of points of $f^{-1}(p)$; one prime (not 0) will be a special case.

Definition. If $R \to S$ is surjective, then the morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ is a *closed immersion of affine schemes.*

Exercise (not on PS). The induced map on sets is an inclusion, and a homeomorphism from $\operatorname{Spec} S$ onto a closed subset of R (with the induced subspace topology).

Example: Galois. Describe Galois for $\mathbb{C}$ over $\mathbb{R}$, or more generally any Galois extension.

1.4. **Morphisms of general schemes.** Recall that with prevarieties, we essentially had two definitions. One was easier to use, involving covers, and the other was more clearly well-defined. You'll see that reflected here.

There is one new twist here. With prevarieties, morphisms were defined as maps of points, and using maps of points, we could pull back functions. We will now need to add that "pullback" to the data of the morphism.

Definition version 1. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two schemes, a *morphism* from X to Y is a continuous map $f : X \to Y$, plus a collection of homomorphisms (one for each open V in Y):

$$f_V^* : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$$

satisfying:

a) *commutes with restriction* Whenever $V_1 \subset V_2$ are two opens, the following diagram commutes:

$$\begin{array}{cccc} f_{V_2}^* : & \mathcal{O}_Y(V_2) & \to & \mathcal{O}_X(f^{-1}(V_2)) \\ & \downarrow \text{res} & & \downarrow \text{res} \\ f_{V_1}^* : & \mathcal{O}_Y(V_1) & \to & \mathcal{O}_X(f^{-1}(V_1)) \end{array}$$

b) If $V \subset Y$ is open and $x \in f^{-1}(V)$, and $a \in \mathcal{O}_Y(V)$, then $a(f(x)) = 0$ implies $f_V^*(a)(x) = 0$. (Remind what that means!)

One has to show that this definition, when X and Y are affine, is just the definition given earlier.

There is a *Definition version 2* involving covering X and Y with affines, which one tends to use in practice. Again, a proof of equivalence is necessary.

Can define *open immersions* and *closed immersions* just as in the case of prevarieties.

1.5. **Scheme-theoretic fibres of a morphism.** (This is a special case of fibre-products, but we haven't discussed fibre-products yet in the category of prevarieties.)

We'll do this by example. Do projection of parabola $x = y^2$ down to x-axis; look where $x \neq 0$ is a closed point; $x = 0$; and generic point.

Exercise: Play around with $\operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}$, corresponding to the inclusion $\mathbb{Z} \to \mathbb{Z}[i]$.