Moduli Spaces in Algebraic Geometry

Math 245A (Winter 2022)

March 11, 2022.

Summary of What We Have Done

A scheme$/k$ "is" a contravariant FUNCTOR

$$(\text{Schemes}/k) \longrightarrow (\text{Sets})$$

Yoneda's Lemma

For a "moduli space" of something, we need a FUNCTOR.

objects $\rightsquigarrow$ families

$$\begin{array}{c} S \\ \downarrow \text{flat} \\ B \end{array}$$

~~finitely presented~~

Noetherian

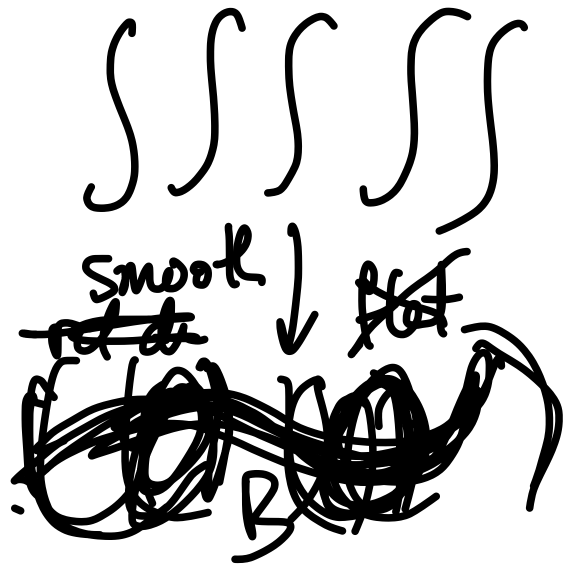
"representable FUNCTOR"

A scheme is not just a FUNCTOR, it is a SHEAF in the Zariski topology (in fact in more...)

Meaning: if X is a ~~scheme~~ FUNCTOR, then maps to X form a sheaf on any Y.

Thus to have a <u>chance</u> of being representable, a moduli FUNCTOR needs to be a SHEAF.

often clear:

$B \mapsto$ 

genus g smooth asymmetric curves

g

Super-useful fact: if $\mathcal{F}$ is a SHEAF that has an "open cover of representable FUNCTORs" then $\mathcal{F}$ is representable.

Moduli space of "functions": $\mathbb{A}^1$.

$\mathbb{P}^n$.

$G(n,k)$ $G(k,n)$

II. The Grassmannian

rank k

version: $0 \to W \to \mathcal{O}^{\oplus n} \to V \to 0$ (later: $\mathbb{P}^{k-1} \hookrightarrow \mathbb{P}^{n-1}$)

$$\begin{array}{ccc} & & | \\ B & \longmapsto & B \end{array}$$

Cover it by $\binom{n}{k}$ open subsets. Fix one I.

$s_1 \wedge \cdots \wedge s_k \neq 0$ in $\det V$.

$\det V$ line bundle

$G(n,k) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$

Plücker

$\mathbb{A}^{k(n-k)}$

$$\begin{tikzcd} \mathcal{O}^{\oplus k} \arrow[d] \arrow[dr, "\sim"] & \\ \mathcal{O}^{\oplus n} \arrow[r] & V \arrow[r] & 0 \end{tikzcd}$$

$\mathcal{O}^{\oplus (n-k)} \to \mathcal{O}^{\oplus k}$.

$n-k$

$\mathbb{A}^{k(n-k)}$ k.

New moduli spaces from old.

Grassmannian bundles. Flag bundles.

III The Hilbert Functor of $\mathbb{P}^n$. The Quot functor.

$$
B \rightsquigarrow \left\{
\begin{tikzcd}
\mathbb{P}^n_B \arrow[d] & X \arrow[l, hook', "\text{cl. emb.}"'] \arrow[dl, "\text{flat}"] \\
B &
\end{tikzcd}
\right.
$$

Input

$$
\mathrm{Quot}\left(
\begin{tikzcd}
\mathbb{P}^n_S \arrow[d] & \mathcal{F} \arrow[l, dash] \\
S &
\end{tikzcd}
\right).
$$

$\mathcal{F} = \mathcal{O}$

Hilbert scheme.

$$
\begin{tikzcd}
& S \\
B \arrow[ur, "\rho"] &
\end{tikzcd}
\qquad
B \rightsquigarrow
\begin{tikzcd}
\rho^*\mathcal{F} \arrow[r] \arrow[d, dash] & Q. \\
\mathbb{P}^n_B \arrow[d] & \\
B. &
\end{tikzcd}
$$

flat over B

Tool: The flattening stratification.

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] & & \mathcal{F}_i \arrow[d, dash] & \text{flat} \\ \mathbb{P}^n_B \arrow[d, dash] & & \mathbb{P}^n_{B_i} \arrow[d, dash] & \\ B & & \bigsqcup B_i \arrow[ll, "\text{loc}"', "\text{closed subschemes.}"] & \end{tikzcd}$$

finitely presented

Then...

First step: $n=0$.

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] \\ B \end{tikzcd}$$

easier.

General case:

set first, then scheme structure, then universal property.

Used $n=0$ case.

Ben Church finished the job.

Cohomology and Base Change

$$\begin{tikzcd}
 & \mathcal{F} \arrow[d, dash] \\
\mathcal{F}' \arrow[d, dash] & \mathbb{P}^n_B \arrow[d, "\pi"] \\
\mathbb{P}^n_{B'} \arrow[ur, "\rho"] \arrow[d, "\tau"'] & B \\
B' \arrow[ur, "\sigma"] &
\end{tikzcd}$$

$\mathcal{F}$ coherent

(need only proper)

$R^i\pi_*\mathcal{F}$

$$\sigma^* R^i\pi_*\mathcal{F} \longrightarrow R^i\tau_*\rho^*\mathcal{F} \quad \text{iso?}$$

$R^i\pi_*\mathcal{F}$ a vector bundle?

Yes after twisting enough

Cohomology and one base change

F flat

Grauert's Theorem (base reduced)

the Cohomology and Base Change Theorem

If $\mathcal{F}$ is flat:

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, dash, dashed] \arrow[d] & \mathcal{F} \\ \operatorname{Spec} A = B & \end{tikzcd}$$

Key idea: Mumford's complex. Over $\operatorname{Spec} A = B$

$$\begin{tikzcd} \cdots \arrow[r] & A^{\oplus} \arrow[r] \arrow[d] & \cdots \arrow[r] & A^{\oplus} \arrow[r] \arrow[d] & A^{\oplus ?} \arrow[r] \arrow[d] & A^{\oplus ?} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^{0} \arrow[r] & \cdots \arrow[r] & \cdots \arrow[r] & C^{n-1} \arrow[r] & C^{n} \arrow[r] & 0 \end{tikzcd}$$

computes $R^i \pi_* \mathcal{F}$ "universally".

Castelnuovo-Mumford Regularity.

$$\begin{tikzcd} \mathcal{F} \arrow[dr, dash] & \\ & \mathbb{P}^n \arrow[d, dash] \\ & \operatorname{Spec} k \end{tikzcd}$$

m-regular: $H^i(X, \mathcal{F}(m-i)) = 0$
$\forall \; i > 0$

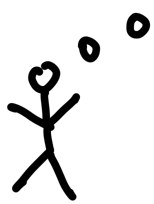
"effective Serre vanishing"

If $\mathcal{F}$ is m-regular, $\mathcal{F}(r)^{r \geq m}$ is gen. by global sections, and has no higher cohomology.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{O}^{\oplus p} \arrow[r] \arrow[d, dash] & Q \arrow[r] & 0 \\ & & \mathbb{P}^n \arrow[d, dash] & p(t) & \\ & & \operatorname{Spec} k & & \end{tikzcd}$$

Theorem:

Given: $n \in \mathbb{Z}^{\geq 0}$, $\quad p \in \mathbb{Z}^{>0}$, $\quad p(t) \in \mathbb{Q}[t]$



"$\mathbb{P}^n$"   "rank"   Hilbert polynomial

Then there is some $m = m(n, p, p(t))$ such that for <u>any</u> coherent sheaf $\mathcal{F} \hookrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus p}$ with Hilbert polynomial $p(t)$, $\mathcal{F}$ is m-regular.

With These we prove:

representability of the Quot scheme and the Hilbert scheme.

$0 \to \mathcal{I} \to \mathcal{O}^{\oplus r} \to Q \to 0$

The idea: (Hilbert scheme). $\mathbb{P}^n$ Fix Hilb pol. $p(t)$.

Find m (regularity).

$X_i \qquad X \overset{\text{closed}}{\underset{\text{subscheme}}{\hookrightarrow}} \mathbb{P}^n \times G$

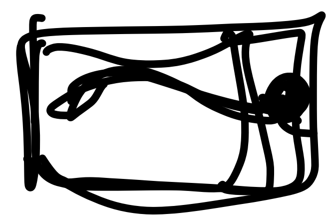
Cut out by degree m polynomials by N polynomials

$\downarrow$

$$Gr\left(\binom{m+n}{m}, N\right)$$

If we fix the Hilbert polynomial, the argument shows that the Quot/Hilbert scheme is ~~quasi~~projective.

$\rightsquigarrow$ projective (work over DVR)



New moduli spaces from old:

"Mor" "Isom" "Aut"

representable

$$\begin{tikzcd} \mathcal{X} \arrow[dr] & \mathcal{Y} \arrow[d] & \underset{\text{representable}}{\mathrm{Isom}(\mathcal{X},\mathcal{Y})} \arrow[dl] \\ & S & \end{tikzcd} \qquad \mathrm{Sch}/S.$$

FUNCTOR/S.

$$B \xrightarrow{e} S \qquad B \mapsto \left\{ \begin{tikzcd} e^*\mathcal{X} \arrow[rr, "\text{iso}"] \arrow[dr] & & e^*\mathcal{Y} \arrow[dl] \\ & B & \end{tikzcd} \right.$$

Theorem

Given:

$$\begin{tikzcd}
\mathcal{L} \arrow[r, dash] & X \arrow[d, "\pi"'] \\
& Y
\end{tikzcd}$$

proper, flat, geometric fibers are connected and reduced (resp., integral)

loc. Noetherian Y

The contravariant FUNCTOR$_Y$ $(\text{Schemes}/Y) \to (\text{Sets})$

$$(Z \to Y) \longmapsto \quad \phi_Z : \pi_Z^* \mathcal{M}_Z \xrightarrow{\sim} \mathcal{L}_Z \qquad
\begin{tikzcd}
\mathcal{L}_Z \arrow[r, dash] & X_Z \arrow[r] \arrow[d, "\pi_Z"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \arrow[r, dash] & \mathcal{L} \\
\mathcal{M}_Z \arrow[r, dash] & Z \arrow[r] & Y &
\end{tikzcd}$$

is a locally closed subfunctor of Y.

(resp., closed)

More moduli spaces (almost) as quotients.

$\mathcal{M}_g \quad g>1$

$C \xrightarrow{|K_C^{\otimes 2}|} \mathbb{P}^N$

~~PGL-bundle~~ $\mathcal{O}(1) \otimes L^{\otimes 2}$

$C \hookrightarrow$ Hilb $\mathbb{P}^N$

closed

$\boxed{}$

$\mathrm{Mor}(C, \mathbb{P}^{\cdot})$

$\mathbb{P}^{\cdot}$

cS

$\mathrm{Pic}^{\circ} C$

C smooth curve $/k$

p^{φ}

$\mathrm{Pic}^{\circ} C \cong \mathrm{Pic}^{d} C$

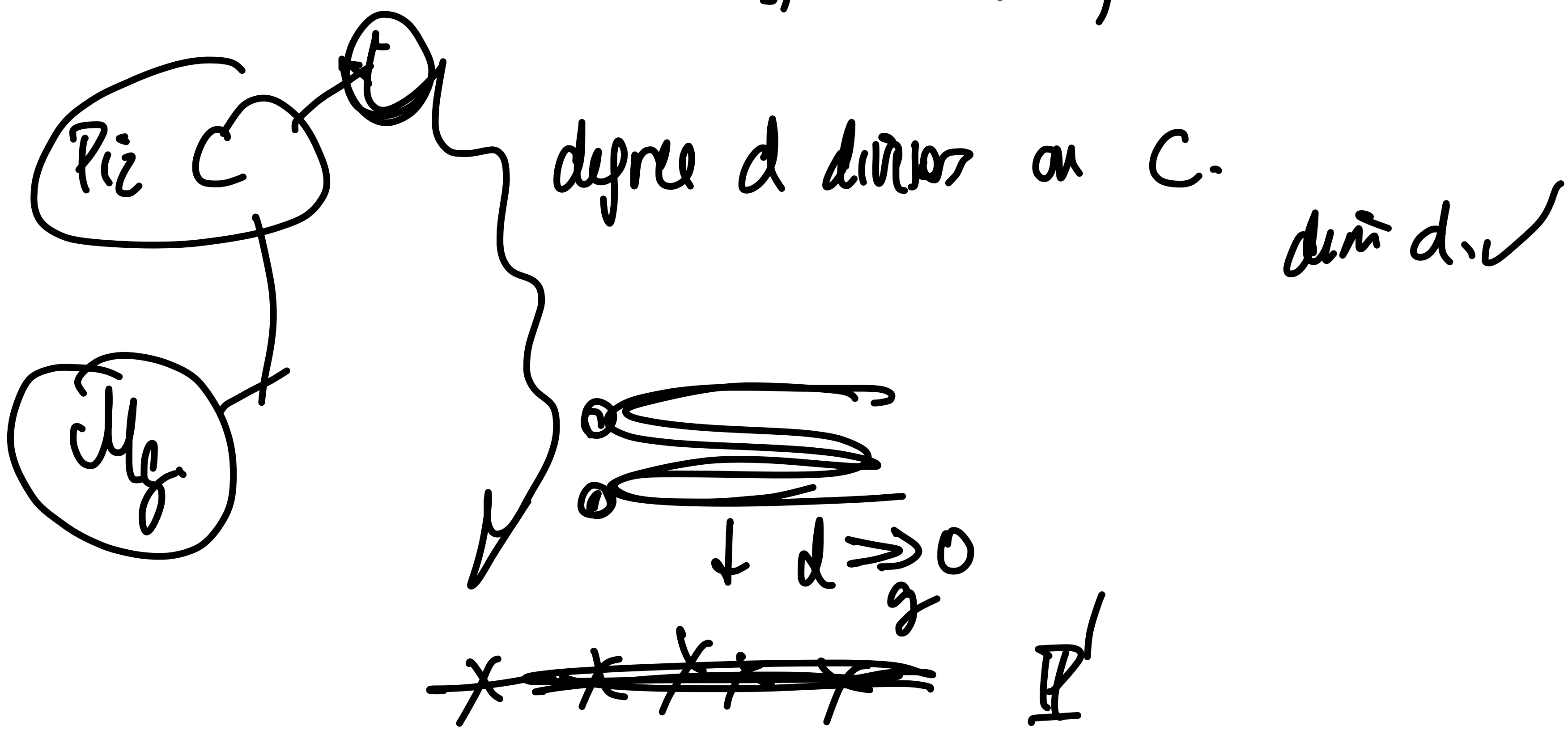
$\mathrm{Pic}^{d} C \xrightarrow{\sim} \mathrm{Pic}^{d+1} C.$

$d \gg 0$

~~Pic~~

What are the dimensions of these spaces?

First version (more handwaving, lower tech)

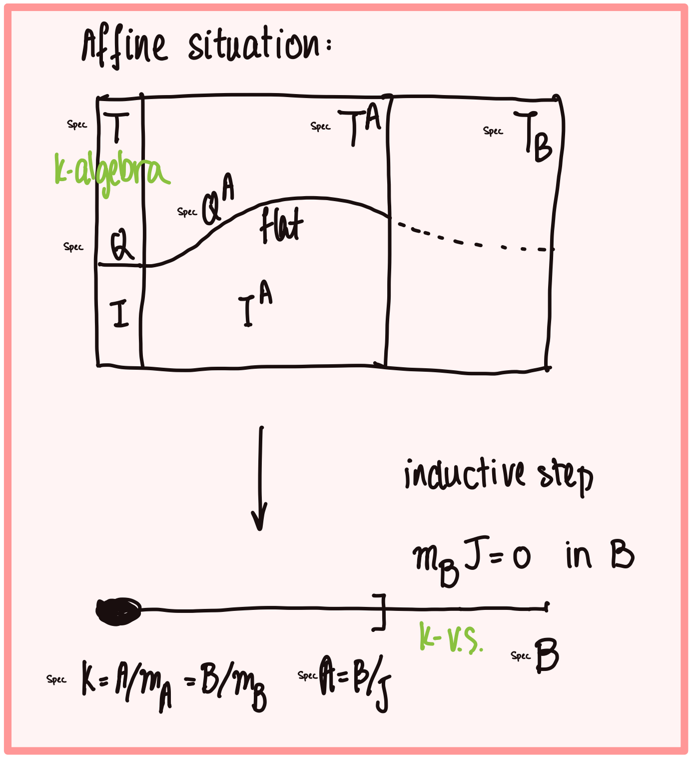


$X \hookrightarrow Y$ both smooth $/k$
projective

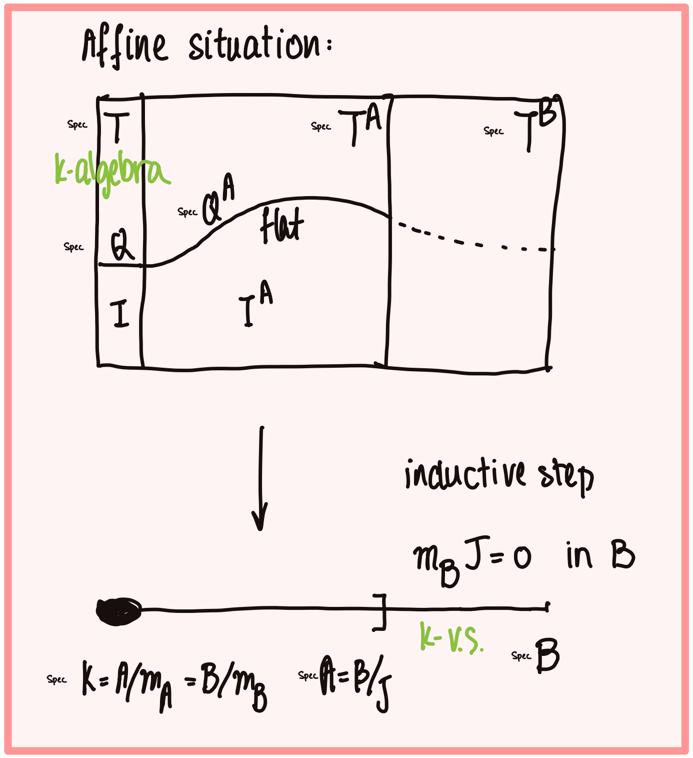
$$T_{\mathrm{Hilb}(X \hookrightarrow Y)} = H^0(X, N_{X/Y})$$

$$\dim \mathrm{Hilb}(X \hookrightarrow Y) \geq h^0(X, N_{X/Y}) - h^1(X, N_{X/Y})$$

Translation: A little bit of deformation theory.



Construction in the argument



Situation

T^B flat / B $\qquad$ Q^A flat / A

Can we extend Q^A to some (flat) Q^B?

"In how many ways?"

Answer: obstruction. $\quad [E^B(Q^A)] \in \mathrm{Ext}^1_T(I, Q\otimes J)$

$$0 \to Q\otimes J \to E^B(Q^A) \to I \to 0$$

Iff it is zero, the choices of Q_B correspond to splittings. They are an affine space over

$$\mathrm{Ext}^0_T(I, Q\otimes J) = \mathrm{Hom}_T(I, Q\otimes J)$$

That's the end of the course —

we covered a lot!

Thanks for coming —

I hope you enjoyed it!