Moduli Spaces in Algebraic Geometry

Math 245 A (Winter 2022)

March 9, 2022.

On Friday, I will tie up any loose ends from today, then go back through the entire course, summarizing what we've done.

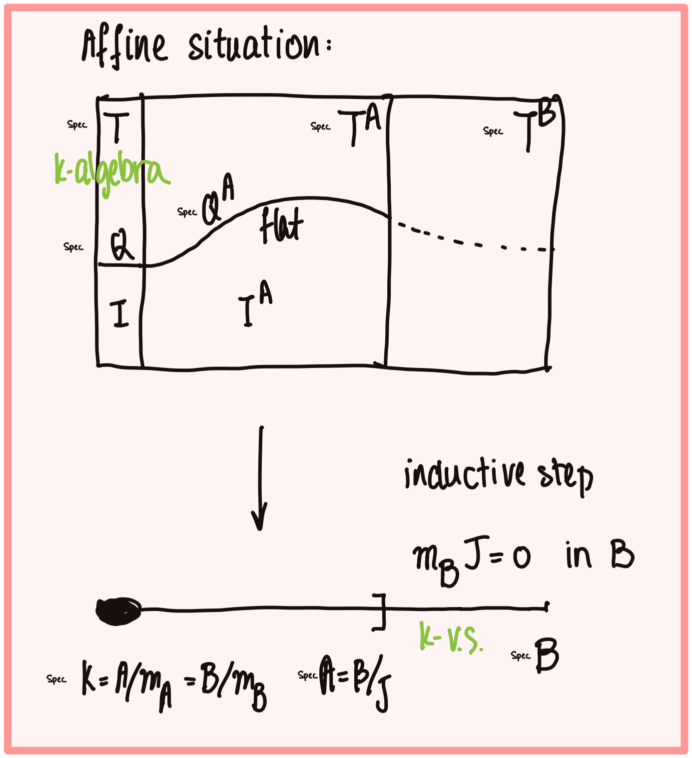
Motivation / Discussion:

$X \hookrightarrow Y$ both smooth $/k$
projective

$$T_{\mathrm{Hilb}(X \hookrightarrow Y)} = H^0(X, N_{X/Y})$$

$$\dim \mathrm{Hilb}(X \hookrightarrow Y) \geq h^0(X, N_{X/Y}) - h^1(X, N_{X/Y})$$

Translation: A little bit of deformation theory.



Situation

T^B flat / B $\qquad$ Q^A flat / A

Can we extend Q^A to some (flat) Q^B?

"In how many ways?"

Answer: obstruction. $[E^B(Q^A)] \in \mathrm{Ext}^1_T(I, Q \otimes J)$

short exact sequence of T-module

$$0 \to Q \otimes J \to E^B(Q^A) \to I \to 0$$

can find a Q^B.

Iff it is zero, the choices of Q_B correspond to splittings. They are an affine space over

$$\mathrm{Ext}^0_T(I, Q \otimes J) = \mathrm{Hom}_T(I, Q \otimes J)$$

$E^B(Q^A) = Q \otimes J \oplus I.$

$X \hookrightarrow Y$

To "globalize" this: $[E^B(Q^A)] = \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{I}_X, \mathcal{O}_X \otimes J)$

Need global extension: $\mathcal{E}xt^1$

Choice of splittings is given by $\mathrm{Hom} = \mathrm{Ext}^0$.

$$\mathrm{Ext}^0_{\mathcal{O}_Y}(\mathcal{I}_X, \mathcal{O}_X \otimes J) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{I}_X, \mathcal{O}_X \otimes J)$$

Version 2 of handwaving: $\underbrace{H^0(\mathcal{E}xt^1) \quad H^1(\mathcal{E}xt^0)}$

$$\mathrm{Ext}^1$$

$H^0(\mathcal{E}xt^0) = \mathrm{Ext}^0.$

$\Gamma(\mathcal{H}om) = \mathrm{Hom}.$

$\mathcal{H}om(\mathcal{I}, \mathcal{O}/\mathcal{I}).$

Let us speculate further.

$X \to Y$

smooth smooth

maybe not closed embedding

$0 \to T_X \to T_Y \to N \to 0$

$X \hookrightarrow Y$

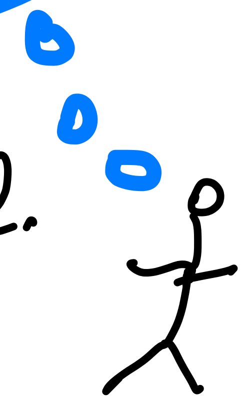
$H^0(N)$ def.

$H^1(N)$ obst.

If $T_X \hookrightarrow T_Y$

Then $\operatorname{coker}(T_X \hookrightarrow T_Y)$.

call that the normal sheaf.



Example:

X C

Y $\mathbb{P}^1$

$N_{C \to \mathbb{P}^1}$ supported at the ramification points of C: dim 0.

$H^1(N) = 0$. unobstructed!

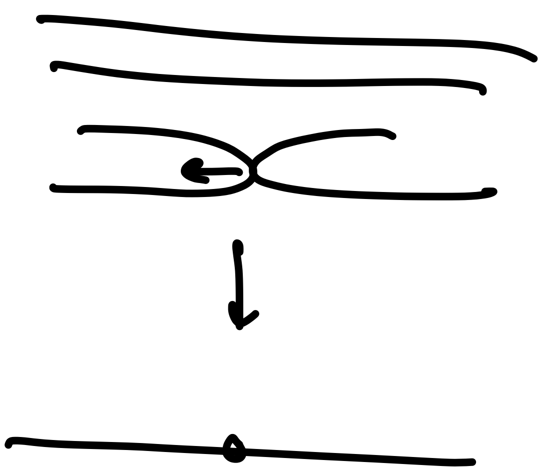
$$X \longrightarrow Y.$$

$$z \longmapsto z^2.$$

<u>Exercise</u> $N_{X \to Y}$ at $z=0$

is skyscraper sheaf of length 1

$\rightsquigarrow$ has 1 section.



What does $\boxed{\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}, \mathcal{O}_X \otimes J)}$ have to do with the normal bundle to X in Y?

$\longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_X, \mathcal{O}_Y/\mathcal{I}_X) \otimes J.$

Check (not instantaneous)

$\mathcal{E}xt^1$ ⤳ $H^1(N_X) \otimes J$

$\longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X) \otimes J.$

$\longrightarrow \mathcal{H}om_{\mathcal{O}_X}(N_X^\vee, \mathcal{O}_X) \otimes J.$

$\longrightarrow = \mathcal{H}om(\mathcal{O}_X, N_X) \otimes J = H^0(N_X) \otimes J.$

Deformations of X if X is not smooth

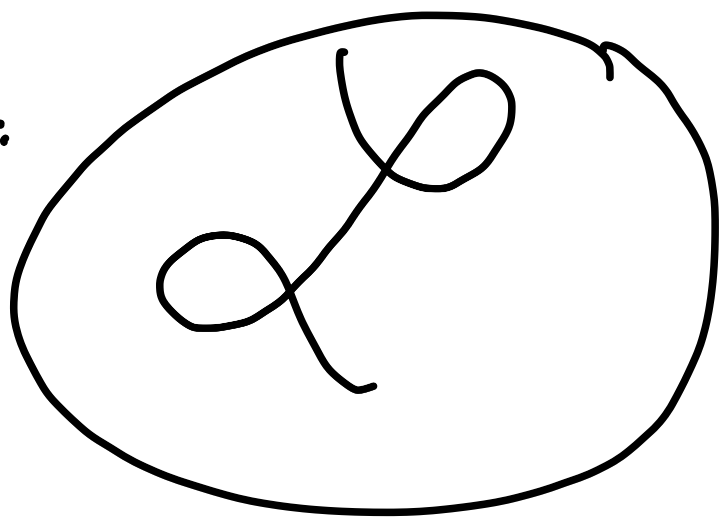
If X were smooth: $H^1(X, T_X)$

If X is <u>not</u> smooth: don't want to discuss T_X.

Ω_X

Expect: $\boxed{\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X).}$

Example:

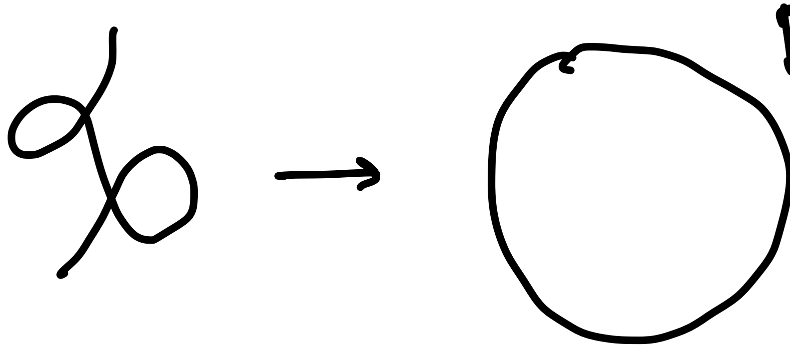


nodal curve.

~~Smooth~~ to smooth case.

$$X \overset{\text{clemb}}{\subset} Y$$

maybe not smooth (curve) — projective smooth



$\mathbb{P}^n$

$$0 \to T_X \to \pi^* T_Y \to N_{X/Y} \to 0.$$

$$0 \to \underset{\text{Aut } X}{H^0(T_X)} \to \underset{\text{Def}(X \to Y)}{H^0(\pi^* T_Y)} \to \underset{\text{Def}(X \to Y)}{H^0(N_{X/Y})} \checkmark$$

$$\to \underset{\text{Def } X}{H^1(T_X)} \to \underset{\text{Ob}(X \to Y).}{H^1(\pi^* T_Y)} \to \underset{\text{Ob}(X \to Y).}{H^1(N_{X/Y})} \to 0 \checkmark$$

"Obx"

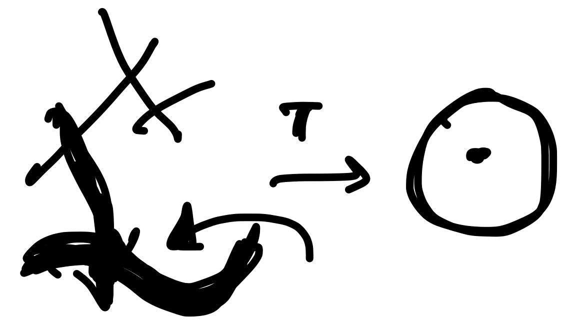
suffices to show

Goal. $= 0$.

$$H^1(C, \pi^* T_{\mathbb{P}^n}).$$

The moduli space of genus 0 stable maps to projective space. is smooth.

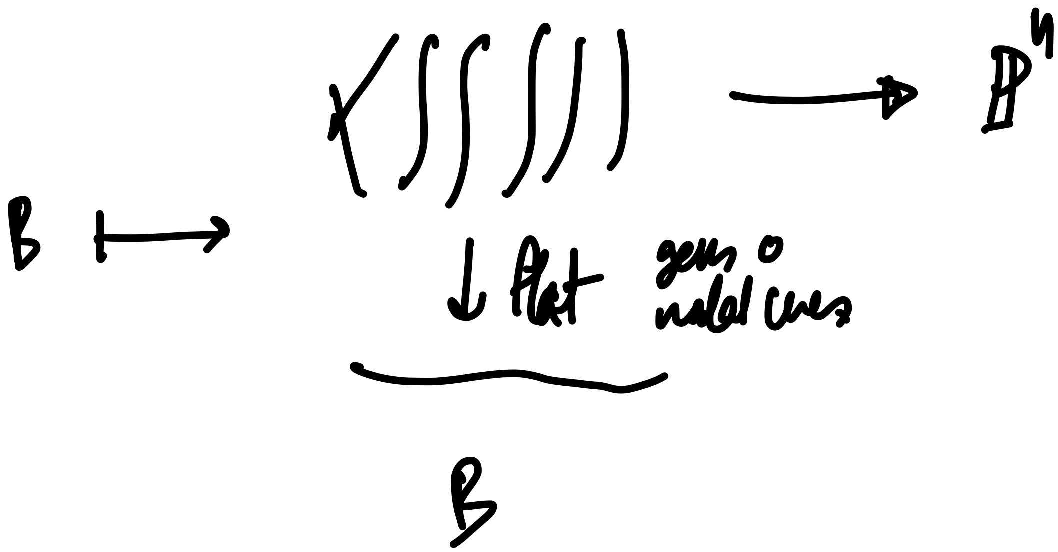
\mathfrak{C}



genus 0 nodal curve.

C

Moduli space of... FUNCTOR.



stability: Aut π finite.

$$\begin{tikzcd} C \arrow[rr] \arrow[dr] & & C \arrow[dl] \\ & \mathbb{P}^n & \end{tikzcd}$$

Any contracted component of C has at least 3 nodes on it.

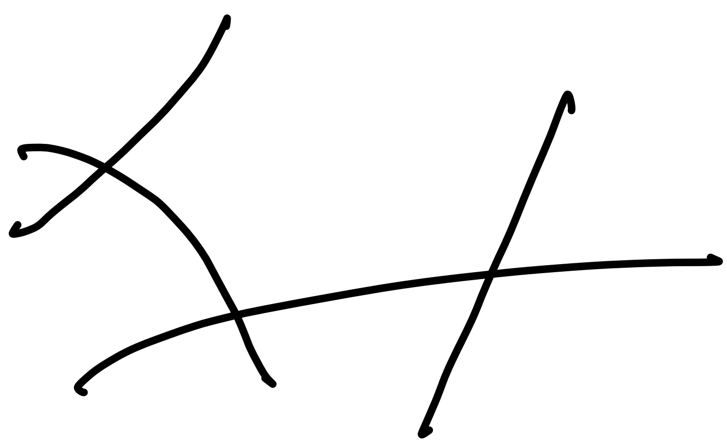
Goal: $H^1(C, \pi^* T_{\mathbb{P}^n}) = 0$ for any stable map

$\pi: C \longrightarrow \mathbb{P}^n$.

$$0 \longrightarrow \pi^* \mathcal{O} \longrightarrow (\pi^* \mathcal{O}(1))^{\oplus (n+1)} \longrightarrow \pi^* T_{\mathbb{P}^n} \longrightarrow 0$$

Goal: $H^1(C, \pi^* \mathcal{O}(1)) = 0$.

genus $C = 0$.

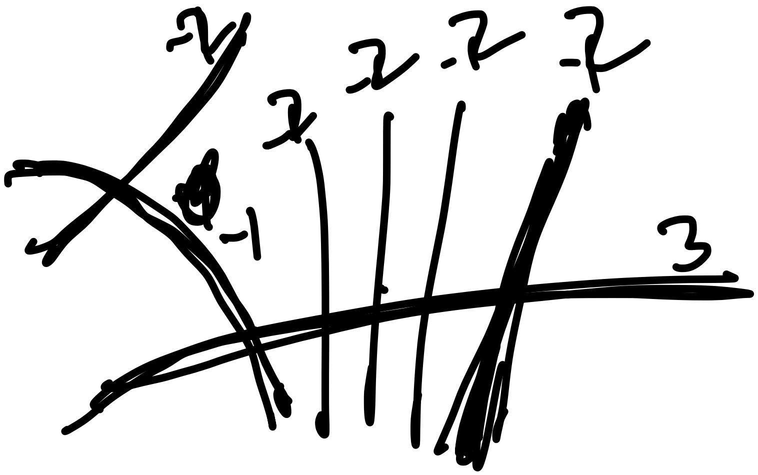


$/ \mathcal{L}$. On any component C_0 of C, $\deg \mathcal{L}|_{C_0} \geq 0$.

> 0 if C_0 has < 3 nodes.

Goal: $H^1(C, \pi^*\mathcal{O}(1)) = 0$. ✓

genus $C = 0$.



$/ \mathcal{L}$. On any component C_0 of C, $\deg \mathcal{L}|_{C_0} \geq 0$.

> 0 if C_0 has <3 nodes.

Serre duality!

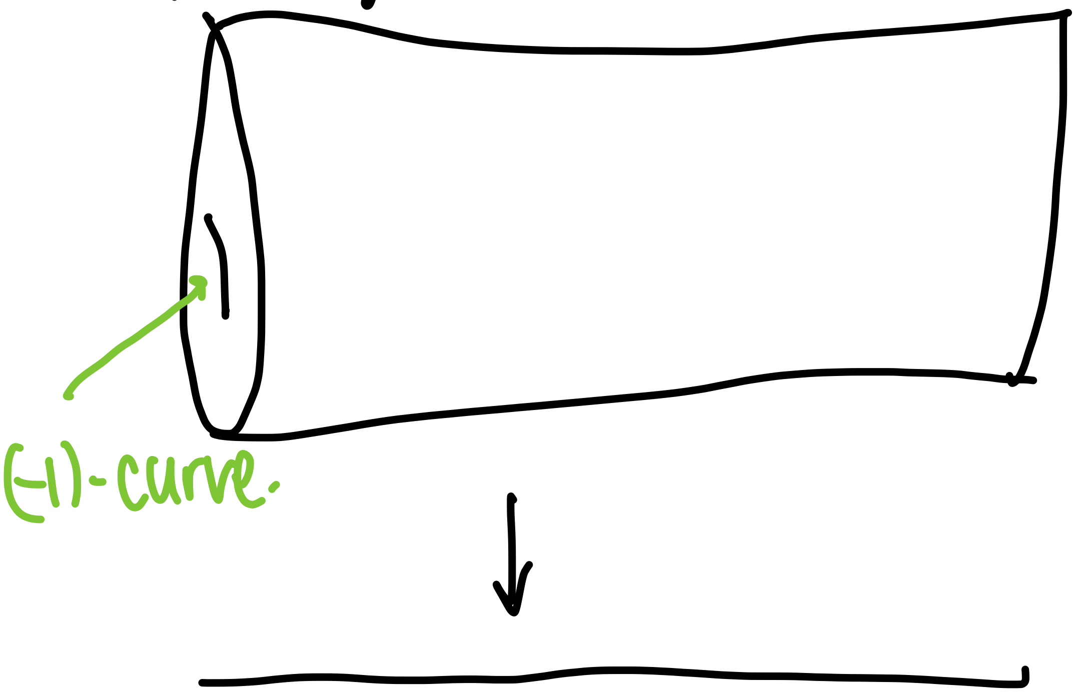
$$h^1(C, \mathcal{L}) = h^0(C, K_C \otimes \mathcal{L}^\vee) = 0$$

On component C_0 with k nodes,

$K_C|_{C_0}$ is $-2 - \deg \mathcal{L}_0 + k$

$\boxed{\text{Kontsevich '94. } \overline{M}_{0,n}(\mathbb{P}^n) \text{ smooth, proper.}}$

Deforming (-1)-curves.



flat family of smooth surfaces