Moduli Spaces in Algebraic Geometry

Math 245 A (Winter 2022)

March 7, 2022.

This week: $X \hookrightarrow Y$ both smooth $/k$
projective

$$T_{\mathrm{Hilb}(X \hookrightarrow Y)} = H^0(X, N_{X/Y})$$

$$\dim \mathrm{Hilb}(X \hookrightarrow Y) \geq h^0(X, N_{X/Y}) - h^1(X, N_{X/Y})$$

Translation: A little bit of deformation theory.

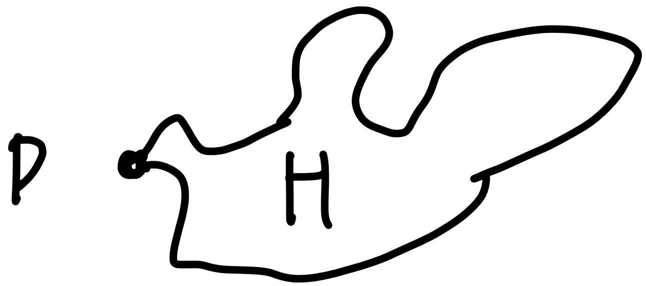
Idea: $H := \mathrm{Hilb}$ is a finite-type k-scheme.

To understand the geometry of H we consider all maps

$$B \longrightarrow H$$

Translation: we understand (indeed, originally <u>defined</u>!) H by the FUNCTOR $(\mathrm{Schemes}_{/k}) \longrightarrow (\mathrm{Sets})$

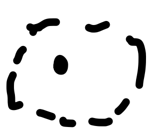
$p := [X \hookrightarrow Y]$ is a (k-valued) closed point of $H = \mathrm{Hilb}$.



To understand the (formal-) local geometry of H near p, we want to understand maps

$$\operatorname{Spec} A \ \left(A \text{ Artin local rings } (A, \mathfrak{m}, k)\right) \longrightarrow H$$

$$[\mathfrak{m}] \longmapsto p$$



Spec A

(Aside: This is another kind of FUNCTOR.)

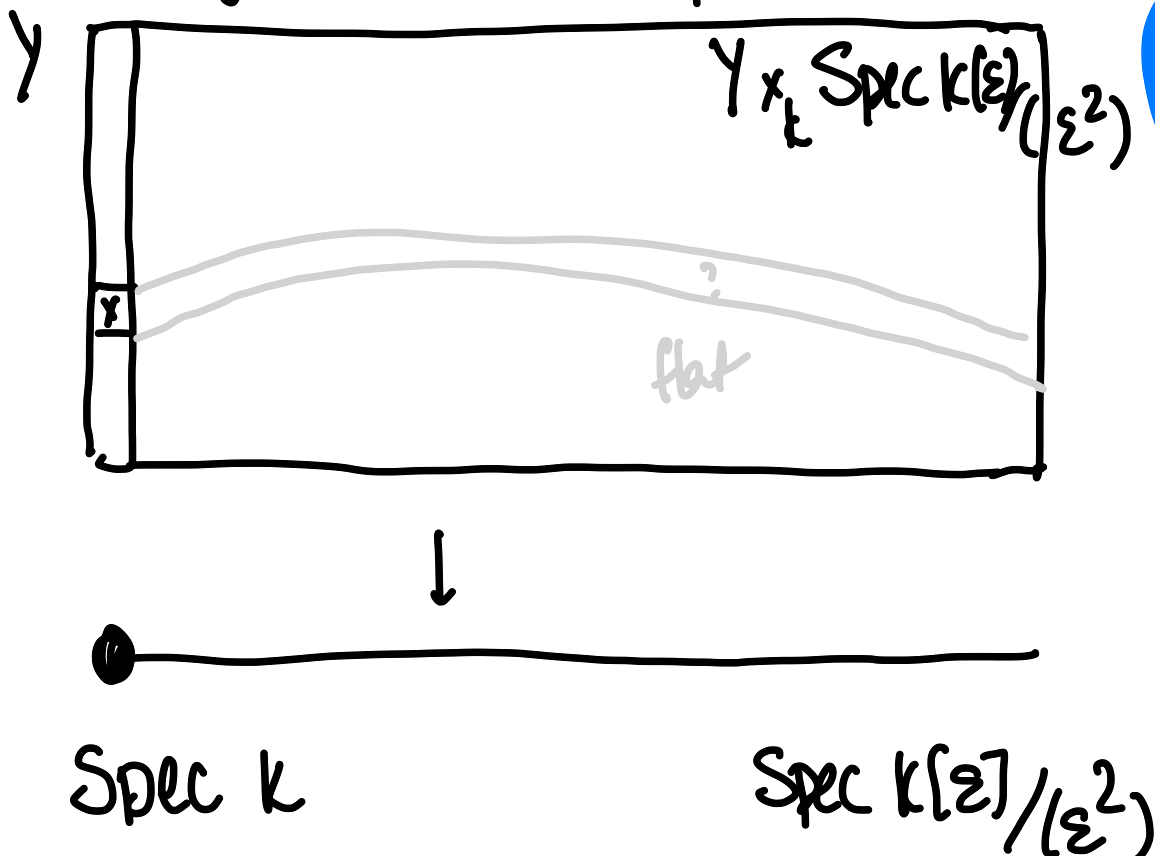
Artin local ring $/_k$:

$k[x_1, \ldots, x_n]/_I$ with one prime ideal.

- Noetherian
- finite length
- $m^l = 0$ for some l
- filtered by powers of m

$$A \supset m \supset m^2 \supset \cdots \supset m^l = (0)$$

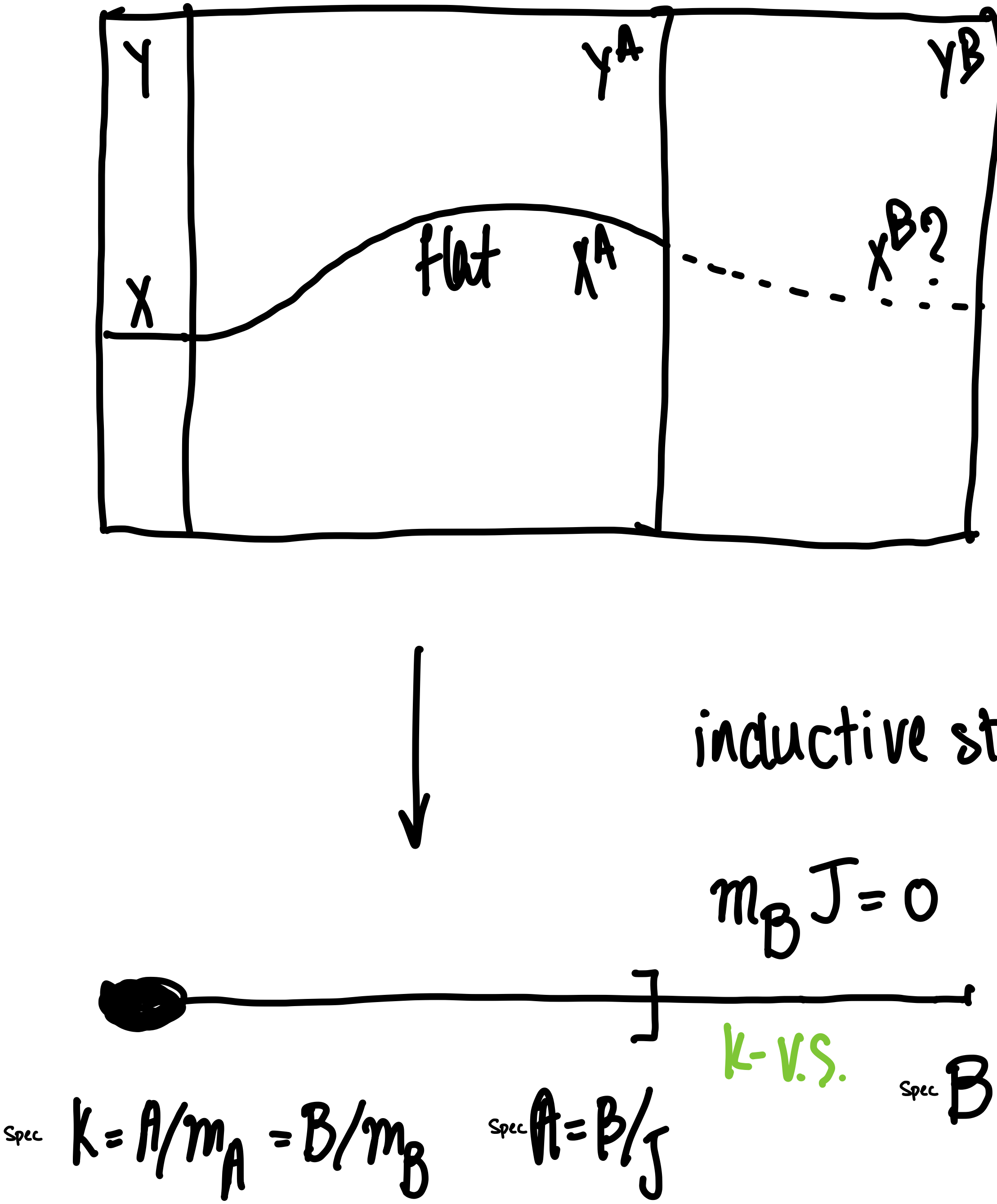
Zariski tangent space question:



Needn't be a product

(Remark on maps as topological spaces)

"Local" geometry of Hilbert scheme



inductive step

$m_B J = 0$ in B

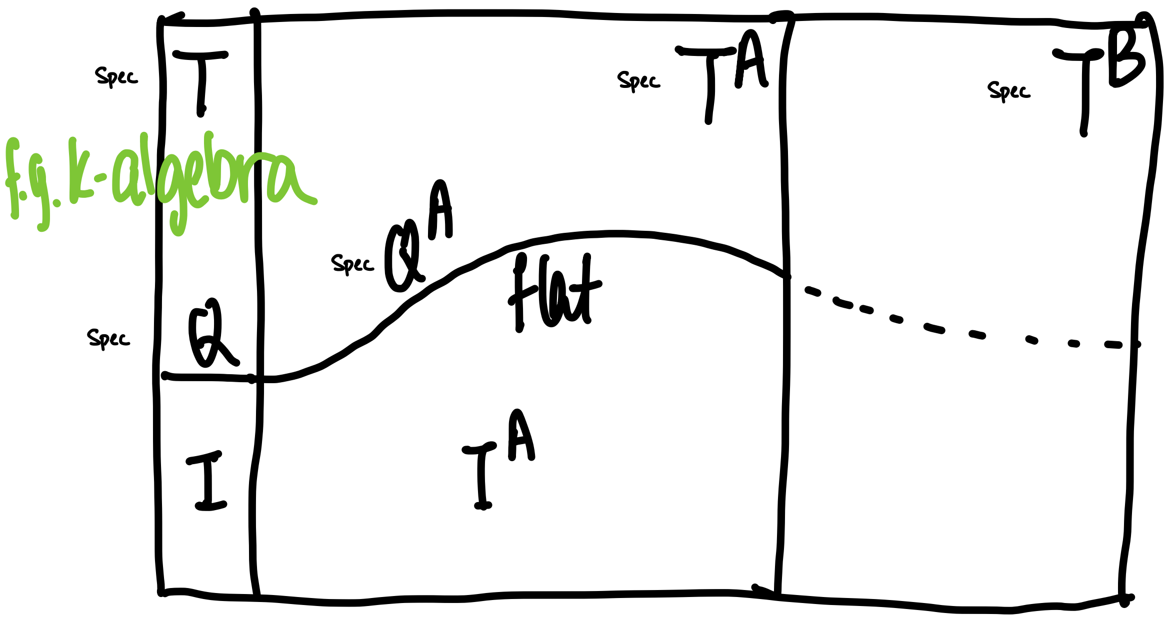
(Remark on maps as topological spaces)

I will follow Kollár's exposition in his wonderful book "Rational curves on algebraic varieties".

Affine situation:

$T^B \otimes_B A = T^A$ etc.

T^B flat$/_B$ etc.
f.g.

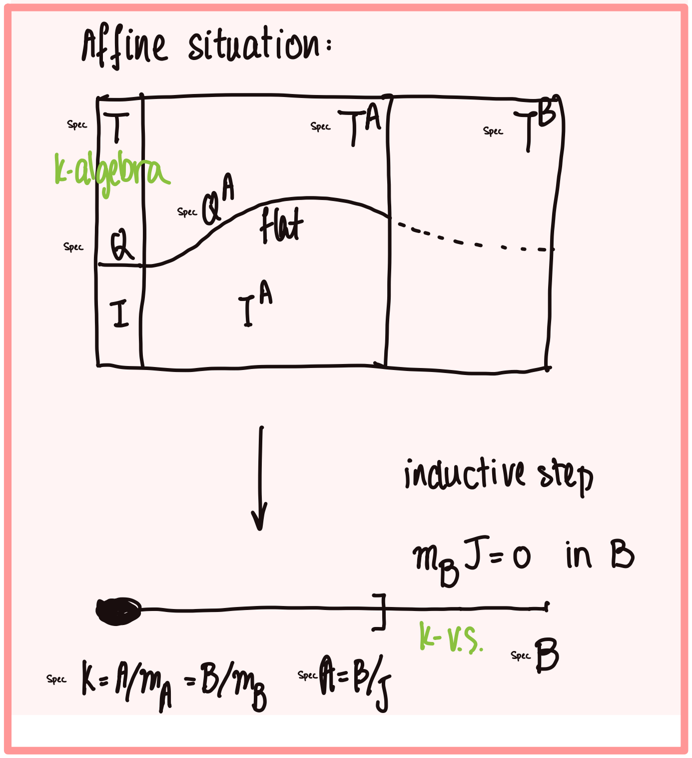


$Q = T/_I$

inductive step

$m_B J = 0$ in B

Spec $k = A/m_A = B/m_B$ Spec $A = B/_J$



Lemma Suppose $\cdots \to F_2 \to F_1 \to I \to 0$ is a free resolution of I (by free T-modules). Then there is a free resolution

$$\cdots \to F_2^A \to F_1^A \to I^A \to 0$$

(of free T^A-modules) "lifting" $(F_\bullet)$.

Translation: $(F_\bullet) = (F_\bullet^A) \otimes_A k$

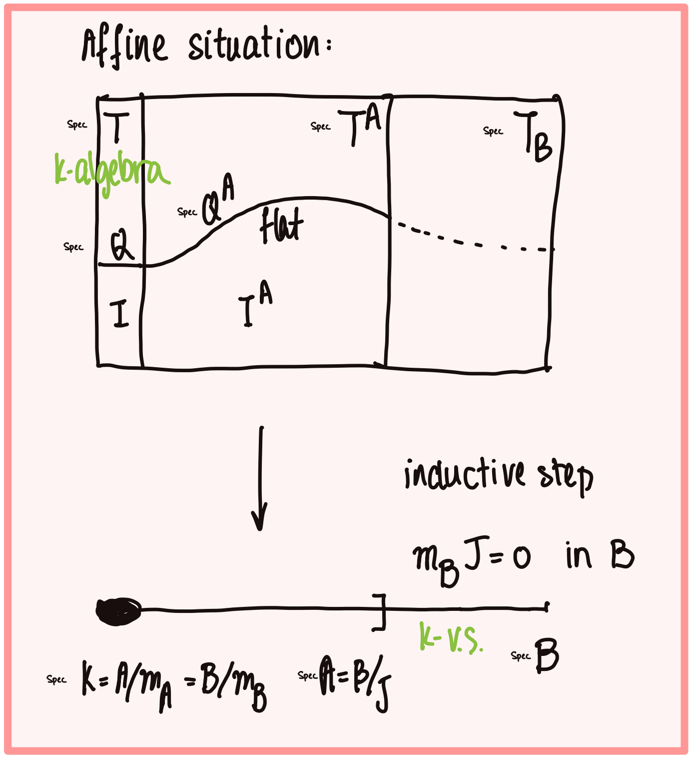
Proof

Q^A flat$/A$ $\Rightarrow$ $I^A \otimes_A k = I$. Reason: $0 \to I_A \to T^A \to Q \to 0$

$\otimes k$:

Choose basis of F_1. Define F_1^A. Kernel of $F_1^A \to I^A \to 0$ is flat.

induct. //



(We've just proved:

Lemma: Free resolutions of I lift to T_A.)

Let's try to find some I_B that works. We'll systematically classify them later.)

free T^B-modules

$\otimes_B A$

$$\begin{tikzcd}
R^B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G^B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & T^B \arrow[r] \arrow[d] & Q^B \arrow[r] & 0 \\
R^A \arrow[r, "r^A"] & G^A \arrow[r, "g^A"] & T^A \arrow[r] & Q^A \arrow[r] & 0
\end{tikzcd}$$

not even a complex

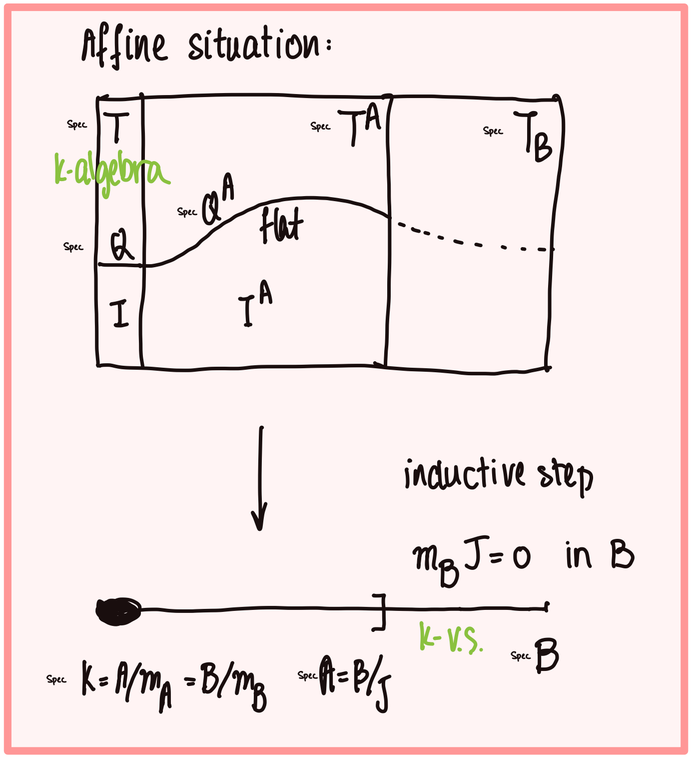
exact

$\underbrace{R^A \quad G^A}_{\text{free } T^A\text{-modules}}$

"generators"

"relations"

$I^B :=$ image of G^B in T^B



$$\begin{tikzcd}
R^B \arrow[r, "r^B"] \arrow[d, "\otimes_B A"'] & G^B \arrow[r, "g^B"] \arrow[d] & T^B \arrow[r] \arrow[d] & Q^B \arrow[r] \arrow[d] & 0 \\
R^A \arrow[r, "r^A"] \arrow[d, "\otimes_A k"'] & G^A \arrow[r, "s^A"] \arrow[d] & T^A \arrow[r] \arrow[d] & Q^A \arrow[r] \arrow[d] & 0 \\
R \arrow[r, "r"] & G \arrow[r, "g"] & T \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

not complex

exact

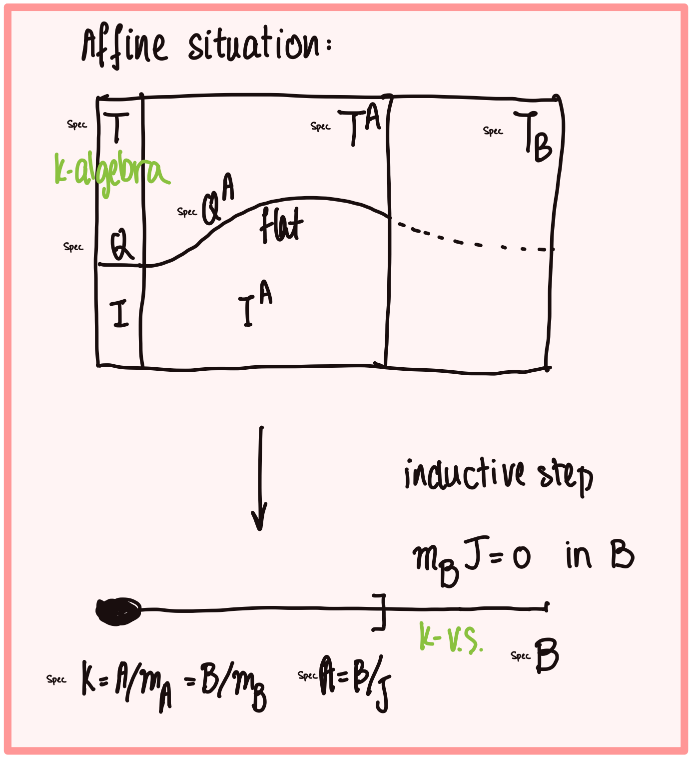
We have an evil plan. Bear with me.

Define map $e(g^B) : R \longrightarrow Q \otimes J$ as follows.

image$\left(R^B \xrightarrow{g^B \circ r^B} T^B \right)$ is $0 \bmod J$ i.e. in $T^B J = T \otimes J$

Thus $\ker\left(R^B \to T^B \right)$ contains $m_B R^B$. $\quad (m_B J = 0)$

so we get a map $R \longrightarrow T \otimes J$ then $\longrightarrow Q \otimes J$.



$$\begin{tikzcd}
R^B \arrow[r, "r^B"] \arrow[d, "\otimes_B A"'] & G^B \arrow[r, "g^B"] \arrow[d] & T^B \arrow[r, "q^B"] \arrow[d] & Q^B \arrow[r] \arrow[d] & 0 \ \text{not complex} \\
R^A \arrow[r, "r^A"] \arrow[d, "\otimes_A k"'] & G^A \arrow[r, "g^A"] \arrow[d] & T^A \arrow[r, "q^A"] \arrow[d] & Q^A \arrow[r] \arrow[d] & 0 \\
R \arrow[r, "r"] & G \arrow[r, "g"] & T \arrow[r, "q"] & Q \arrow[r] & 0
\end{tikzcd}$$

exact

Define map $e(g^B) : R \to Q \otimes J$ as follows.

image($R^B \xrightarrow{g^B \circ r^B} T^B$) is 0 mod J i.e. in $T^B J = T \otimes J$

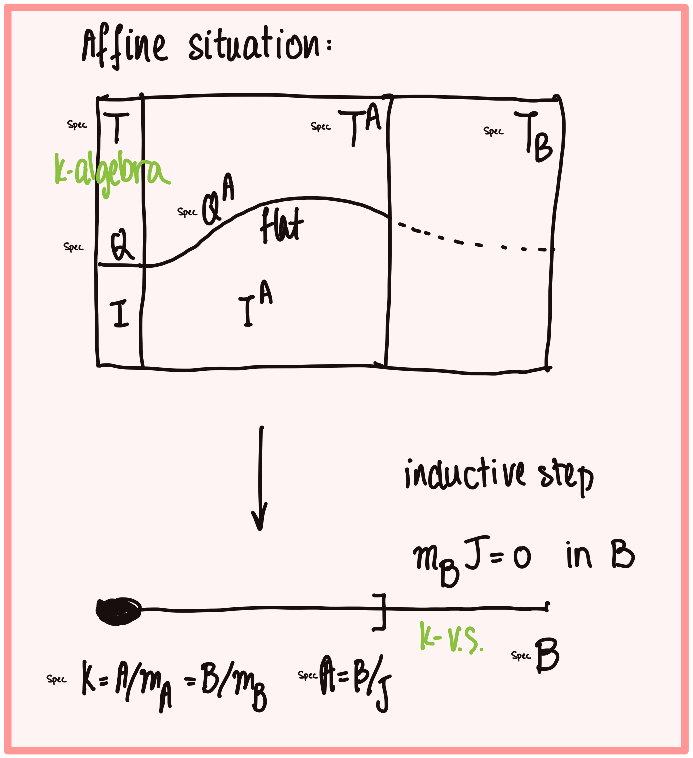
Thus ker ($R^B \to T^B$) contains $m_B R^B$. ($m_B J = 0$)

so we get a map $R \to T \otimes J$ then $\xrightarrow{q} Q \otimes J$.

This is independent of r^B: if we take a different lift $\tilde{r}^B$ of r^A, $\tilde{r}^B = r^B + \varphi^B$ ($\varphi^B \in \mathrm{Hom}(R^B, G \otimes J)$),

then:
$$q \circ g^B \circ \tilde{r}^B - q \circ g^B \circ r^B$$
$$= q \circ g^B \circ \varphi^B \underset{\text{TRICKY}}{=} q \circ [\underbrace{(g \otimes id_J) \circ \varphi^B}_{\subset I \otimes J}] = 0.$$

//



$$\begin{tikzcd}
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\end{tikzcd}$$

not complex

exact

Defined:

$e(g^B): \; R \longrightarrow Q \otimes J$

"$q \circ g^B \circ r^B$"

Theorem The following are equivalent:

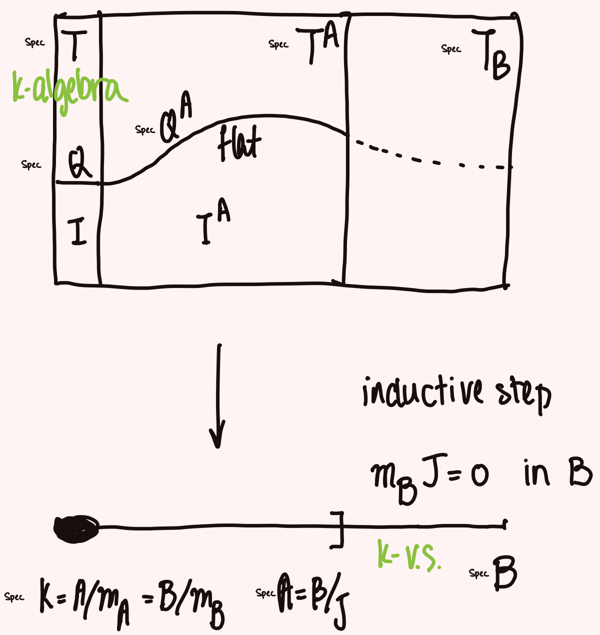
(i) Q_B is flat over B.

(ii) $e(g^B) = 0$

(iii) $g^B \circ r^B = 0$ (for suitably chosen r^B)

I think I'll omit this proof.

Affine situation:



$$\begin{tikzcd}
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exact

Defined:

$e(g^B)$: $R \longrightarrow Q \otimes J$

"$q \circ g^B \circ r^B$"

0 if flat

Theorem The following are equivalent:

(i) Q_B is flat over B.

(ii) $e(g^B) = 0$

(iii) $g^B \circ r^B = 0$ (for suitably chosen r^B)

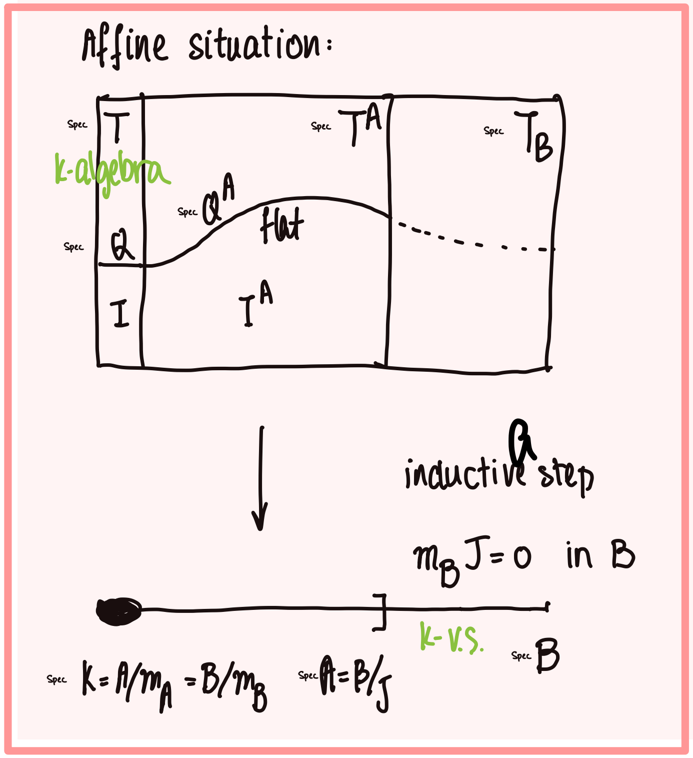
Next: define $E(g^B) := \operatorname{coker}\left(R \xrightarrow{e(g^B)+r} (Q \otimes J \oplus G)\right)$

Lemma We have an exact sequence

$$0 \to Q \otimes J \to E(g^B) \to I \to 0$$

(hence an element of $[E(g^B)] \in \operatorname{Ext}^1_T(I, Q \otimes J)$)

(if $e(g^B) = 0$, $E(g^B) = Q \otimes J \oplus I$.)



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Defined:

$e(g^B) : R \to Q \otimes J$

"$q \circ g^B \circ r^B$"

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(hence an element of $[E(g^B)] \in \mathrm{Ext}^1_T(I, Q \otimes J)$)

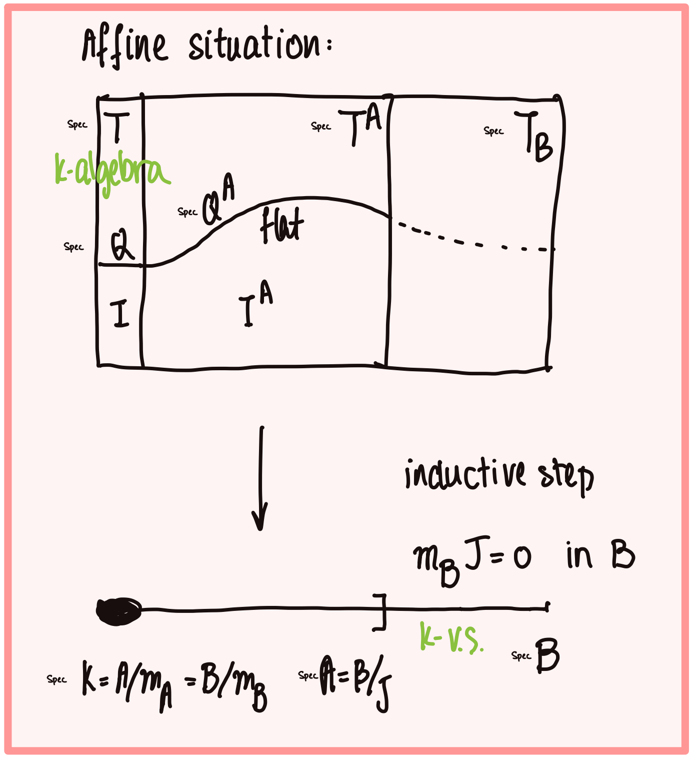
Proof Exactness here is clearer: $Q \otimes J \to E(g^B) \to I \to 0$

To show $Q \otimes J \to \mathrm{coker}\left(R \xrightarrow{e(g^B)+r} (Q \otimes J \oplus G)\right)$

is injective, we need to show that $\ker(e(g^B)) \supset \ker(r)$. (explain)

So suppose $a \in \ker(r)$. Lift it to $a^A \in \ker(r^A)$

(our lifting lemma). Lift to $a^B \in R^B$. $r^B(a^B) \in JG^B$. (explain) //



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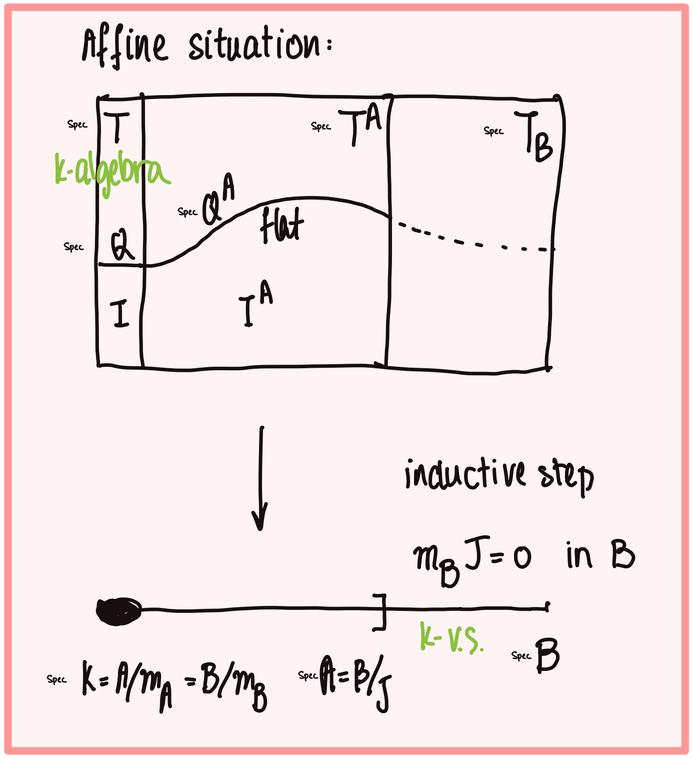
(hence an element of $[E(g^B)] \in \operatorname{Ext}^1_T(I, Q \otimes J)$)

For the sake of your sanity, I'll state the next few things without proof.

Any different lifting $\tilde{g}^B : G^B \to T^B$ gives the same extension in $\operatorname{Ext}^1_T(I, Q \otimes J)$.

Call this class $E^B(Q^A) \in \operatorname{Ext}^1_T(I, Q \otimes J) = \operatorname{Ext}^1_T(I, Q) \otimes_k J$

It depends only on the data over A! It is called the obstruction.



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Defined:

$e(g^B): R \to Q \otimes J$

" $q \circ g^B \circ r^B$ "

$E(g^B) := \operatorname{coker}\left(R \xrightarrow{e(g^B)+r} (Q \otimes J \oplus G)\right)$

$0 \to Q \otimes J \to E(g^B) \to I \to 0$

obstruction.

$$E^B(Q^A) \in \operatorname{Ext}^1_T(I, Q \otimes J)$$

Theorem

(i) $E^B(Q^A) = 0$ iff there is a suitable q^B for which Q^B is flat.

(ii) In this case, we have non-canonical affine-linear isomorphisms

$$\{\text{flat quotients } Q^B\} \xleftrightarrow{\sim} \text{splittings of } 0 \to Q\otimes J \to E(g^B) \to I \to 0 \longleftrightarrow \operatorname{Hom}_T(I, Q \otimes J)$$

Nice to know:

All of these statements are independent of the choice of presentation $R^A \to G^A \to I^A \to 0$.