Moduli Spaces in Algebraic Geometry

Math 245 A (Winter 2022)

March 4, 2022.

From last day...

Fix a curve C over a field k.

For $d \geqslant 2g+1$, we "defined" $\mathrm{Pic}^d C$ as follows

Our thinking: if $\deg \mathcal{L} = d$,

$h^1(C, \mathcal{L}) = 0$ (just ask why)

$h^0(C, \mathcal{L}) \overset{RR}{=} d-g+1$

These sections give a closed embedding:

$$C \xrightarrow[\text{cl. emb.}]{|H^0(C,\mathcal{L})|} \mathbb{P}^{d-g}$$

The image is a degree d genus g curve.

Hilb. pol:

$p(t) = dt + (1-g)$

Consider

$$\begin{tikzcd}[column sep=small]
 & & & & \mathbb{P}^{d-g}\times \mathrm{Isom} \arrow[ddr] & \rho^* U \arrow[l, hook'] \arrow[dd] & C\times \mathrm{Isom} \arrow[l, "\sim"'] \arrow[ddl] \\
 & & & & & & \\
 & \mathbb{P}^{d-g}\times \mathrm{Hilb} \arrow[r, hook, "\text{cl. emb.}"] \arrow[d] & U \arrow[dl] & C\times \mathrm{Hilb} \arrow[d] & & \mathrm{Isom}_{\mathrm{Hilb}}(U, C\times \mathrm{Hilb}) \arrow[dll, "\rho"] \arrow[dr, hook, "\text{open}"] & \\
\mathrm{Aut}\,\mathbb{P}^{d-g} \arrow[r, phantom, "\circlearrowright"] & \mathrm{Hilb}_{p(t)} & & \mathbb{P}^{d-g} & & & \mathrm{Hom}(C, \mathbb{P}^{d-g})
\end{tikzcd}$$

not in hyperplane.

$$\mathrm{Pic}^d_g = \mathrm{Isom}_{\mathrm{Hilb}}(U, C\times \mathrm{Hilb})^{\text{non-deg.}} \big/ \mathrm{Aut}\,\mathbb{P}^{d-g}$$

or $\mathrm{Isom}_{\mathrm{Hilb}}(U, C\times \mathrm{Hilb})^{\text{non-deg.}} \to \mathrm{Pic}^d_g$ is an $\mathrm{Aut}\,\mathbb{P}^{d-g}$-bundle.

Another definition of $\mathrm{Pic}^d C$:

$k = \bar{k}$ (?)

Fix a point $p \in C$.
k-valued

FUNCTOR

$$ {}_p\mathrm{Pic}^d C : (\mathrm{Schemes}/k) \longrightarrow (\mathrm{Sets})$$

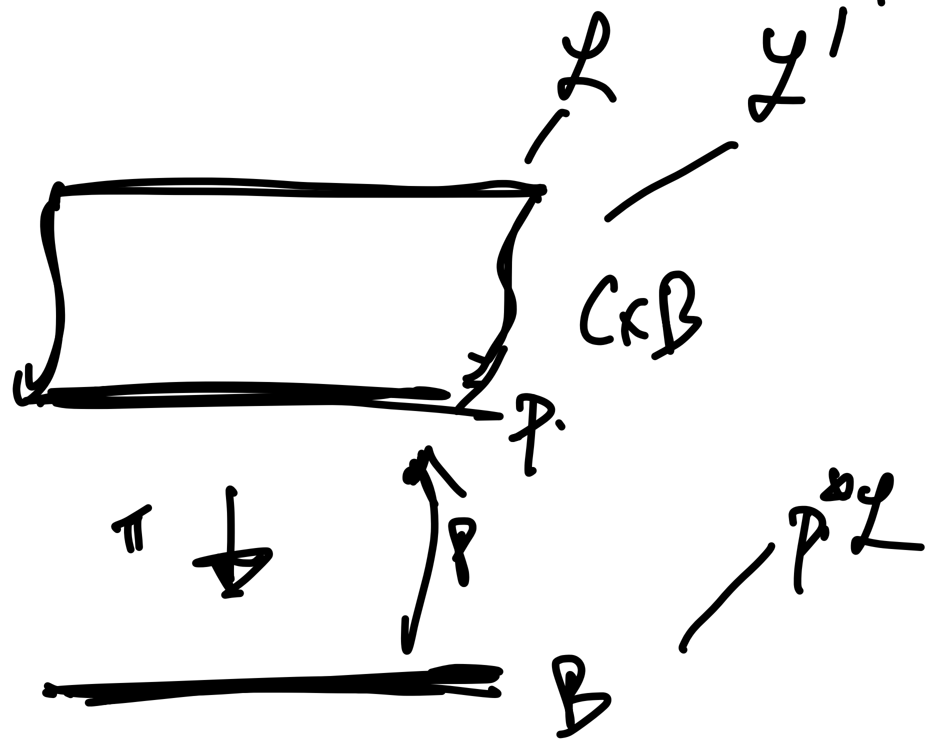
$$B \longmapsto \begin{tikzcd} & \mathcal{L} \arrow[dl, dash] \\ C \times_k B \arrow[d, shift right] & \\ B \arrow[u, shift right, "p."'] & \end{tikzcd}$$

$\mathcal{L}$ degree d on fibers.

trivialized @ p $\quad p^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_B$

Proposition: This is independent of p!

Idea



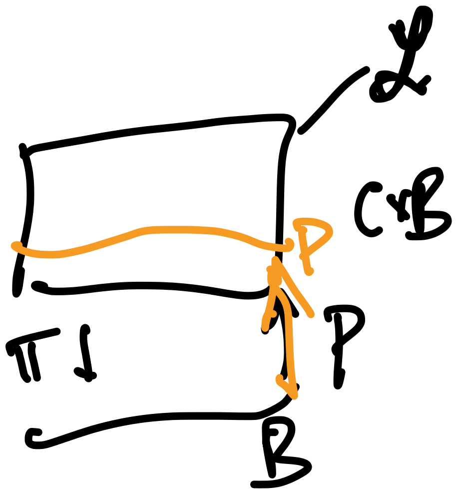
$$\boxed{\mathcal{L}' = \mathcal{L} \otimes \pi^* (p^* \mathcal{L})^\vee}$$

$p^* \mathcal{L}' \xrightarrow{\sim} \mathcal{O}$; fiber by fiber. $\mathcal{L}_b \cong \mathcal{L}'_b$

$$(\text{Schemes}/k) \longrightarrow \text{Sets}$$

My functor: line bundles trivialized at $p \in C$

$B \longmapsto$



$\phi : p^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_B$

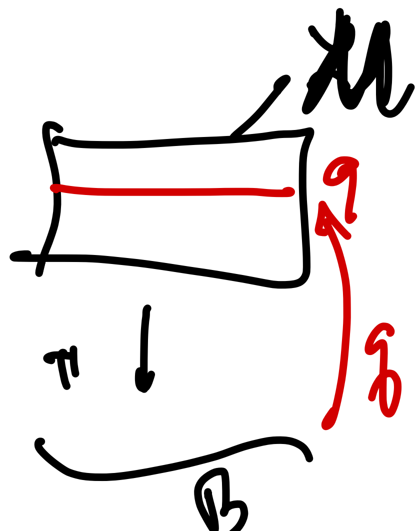
But: $q^*\mathcal{L} \not\cong \mathcal{O}_B$

$$\mathcal{M} := \mathcal{L} \otimes \pi^*(q^*\mathcal{L})^\vee$$

$$q^*\mathcal{M} = q^*\mathcal{L} \otimes \boxed{q^*\pi^*} q^*\mathcal{L}^\vee$$
$$= q^*\mathcal{L} \otimes q^*\mathcal{L}^\vee$$
$$= \mathcal{O}$$

Yon functor trivialized at $q \in C$

$B \longmapsto$



$\tau : q^*\mathcal{M} \xrightarrow{\sim} \mathcal{O}_B$

esp.

What if k is not algebraically closed?

Aside: What if C has no k-points?

Example: $k = \mathbb{R}$ $\quad C = x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$

$g = 0.$

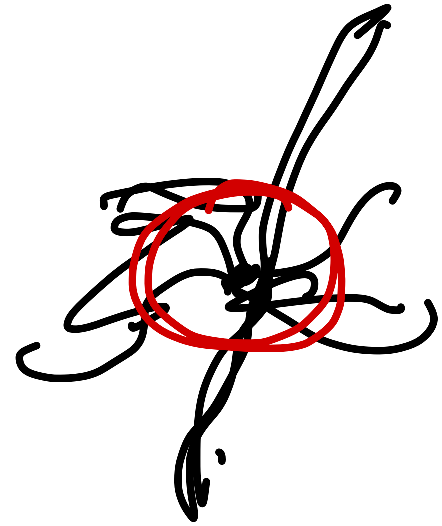
$d = 1$

$$\boxed{\mathrm{Pic}^1 C = \text{point.}}$$

We now discuss how to get a hold of the infinitesimal structure of the Hilbert / Quot scheme.
I'll do some examples taking the deformation theory as a black box, and we will conclude the course with a proof of an important example of the black box.

Example $X \overset{\text{closed embedding}}{\hookrightarrow} Y$ both smooth over k

projective

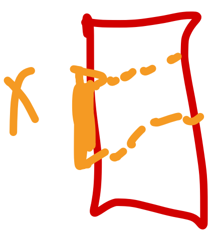


This is a point of $\mathrm{Hilb}\, Y$.

a k-valued point.

What is the moduli-theoretic interpretation of the tangent space to $\mathrm{Hilb}\, Y$ at $[X \hookrightarrow Y]$?

$Y \times \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$



$$
\begin{tikzcd}
\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \arrow[r] & \mathrm{Hilb}\, Y \\
\operatorname{Spec} k \arrow[u, hook] \arrow[r] & {[X \hookrightarrow Y]} \arrow[u, hook]
\end{tikzcd}
$$

give Z.

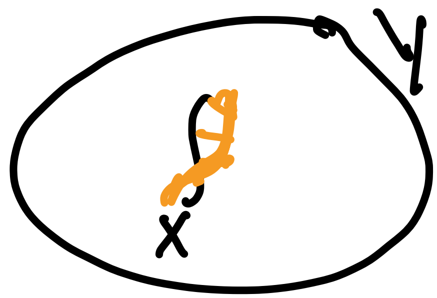
tangent space.

given: $\circ \to \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$

$X \hookrightarrow Y$ / field k.

Deep Fact: The tangent space, with its vector space structure ($/k$), is $H^0(X, N_{X/Y})$.

Picture:



Deeper fact: If $H^1(X, N_{X \to Y}) = 0$, then the Hilb Y is smooth at this point (of dimension $h^0(X, N_{X/Y})$.)

Proof: maybe next week.

Deepest Fact: The dimension of the Hilbert scheme over a field k at a point $X \hookrightarrow Y$ is at least

$$h^0(X, N_{X/Y}) - h^1(X, N_{X/Y}).$$

Proof: maybe next week.

Summary:

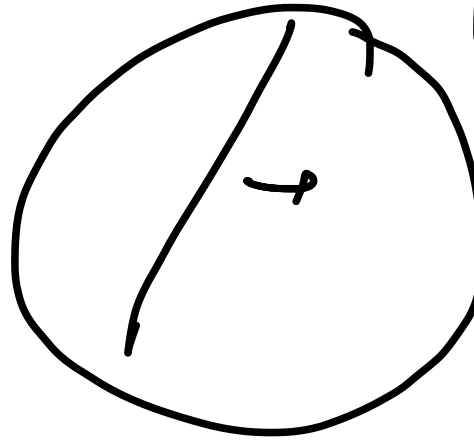
$$h^0(N) - h^1(N) \leq \dim \mathrm{Hilb} \leq h^0(N) = \dim T_{\mathrm{Hilb}}$$

smooth if equality

Examples:

Line in the plane

line L $\mathbb{P}^2$



$N_{L/\mathbb{P}^2} = \mathcal{O}(1)$

Hilb. $\mathbb{P}^{2\vee}$

$h^0(L, \mathcal{O}(1)) = 2$

$h^1(L, \mathcal{O}(1)) = 0$

Example:

Curve C : smooth projective integral / k

genus g, $\pi : C \hookrightarrow \mathbb{P}^{d-g}$, degree d.

cl. emb. $\quad \mathbb{P}^{d-g} = \mathbb{P}$

$d \gg 0$.

$$0 \to T_C \to \pi^* T_{\mathbb{P}^{d-g}} \to N_{C/\mathbb{P}} \to 0$$

$H^0(N) \quad H^1(N)$

$\pi : C \to \mathbb{P}^{d-g}$

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus(d-g+1)} \to \pi^* T_{\mathbb{P}^{d-g}} \to 0$$

$$0 \to H^0(C,\mathcal{O}) \to H^0(C,\mathcal{L})^{\oplus d-g+1} \to H^0(\pi^* T_{\mathbb{P}}) \to H^1(C,\mathcal{O}) \to H^1(C,\mathcal{L})^{\oplus d-g+1} \to H^1(\pi^* T_{\mathbb{P}}) \to 0$$

$$h^1(C, \pi^* T_{\mathbb{P}^{d-g}}) = 0$$

$$h^0(C, \pi^* T_{\mathbb{P}^{d-g}}) = (d-g+1)(d-g+1) + g - 1$$

$$0 \to T_C \to i^* T_{\mathbb{P}^{d-g}} \to N_{X/\mathbb{P}} \to 0$$

Long exact sequence:

$(d-g+1)(d-g+1)+g-1$

$$0 \to H^0(T_C) \to \boxed{H^0(\pi^* T_{\mathbb{P}^{d-g}})} \to H^0(N)$$

$$\to H^1(T_C) \to H^1(\pi^* T_{\mathbb{P}}) \to H^1(N) \to 0$$

$$\boxed{(d-g+1)^2 + (4g-4)}$$

$H^1(C, N_{C/\mathbb{P}}) = 0$

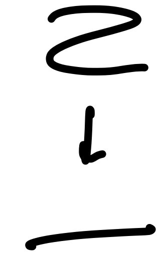
degree $2-2g$

$H^0(C, N_{C/\mathbb{P}}) = (d-g+1)^2 + g - 1 - \underbrace{\chi(T_C)}_{2-2g-g+1}$

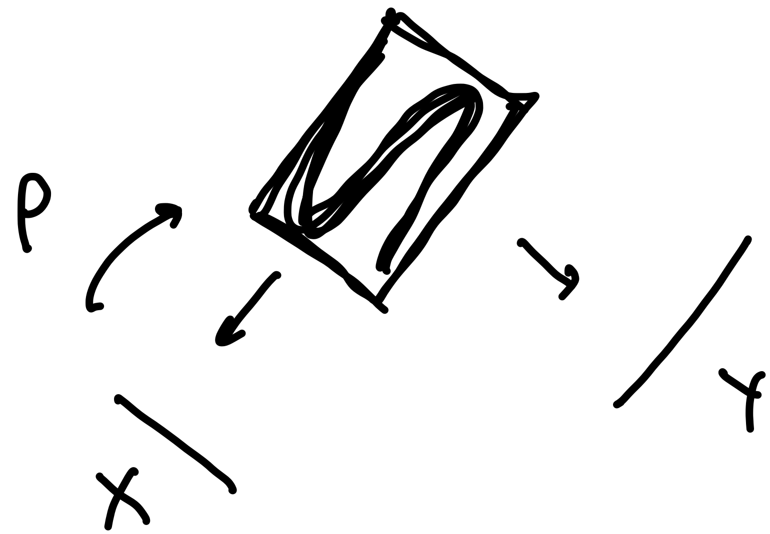
$= (d-g+1)^2 + (g-1) + 3g-3.$

$\dim \mathcal{M}_g = \underline{\dim \mathrm{Hilb}} - \underline{\dim \mathrm{Aut}(\mathbb{P}^{d-g})} = (d-g+1)^2 + 4g - 4.$

What's wrong? $\boxed{4g-3} = -((d-g+1)^2 - 1)$

How about deforming maps $X \xrightarrow{\pi} Y$ (smooth + proj!) e.g. $\begin{array}{c} C \\ \downarrow \\ \mathbb{P}^1 \end{array}$ 

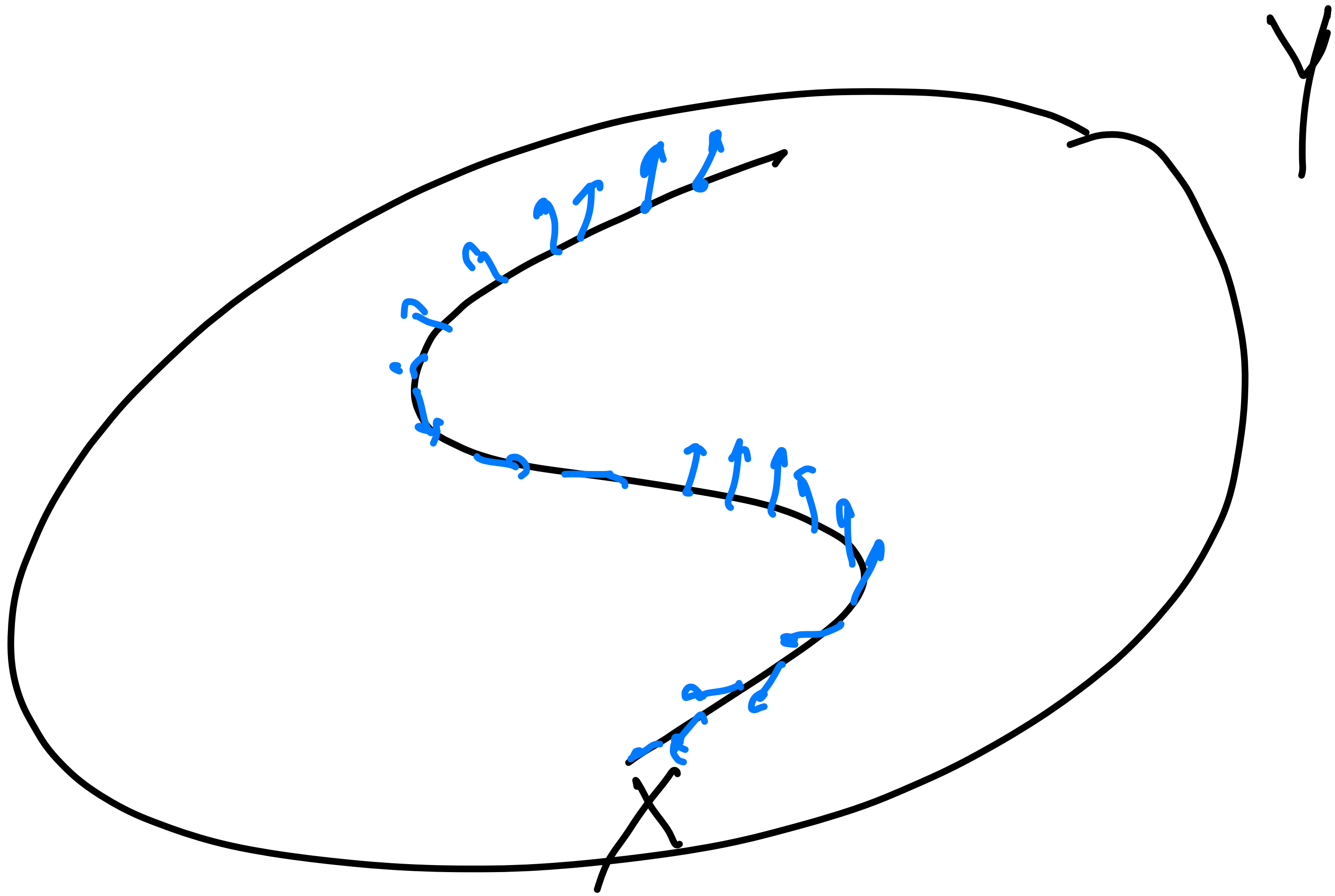
The Hilbert scheme in question: $X \xhookrightarrow{\Gamma_\pi} X \times Y$



$$0 \to T_X \to \rho^*(T_{X\times Y}) \to N \to 0$$

$$0 \to T_X \to T_X \oplus \pi^* T_Y \to N \to 0$$

Answer: $\pi^* T_Y = N$.


Y
X

Let us deform maps

$$C \xrightarrow{\pi} \mathbb{P}^{d-g} \quad \text{degree } d \quad (\text{with } C \text{ fixed})$$

$\mathrm{Hom}(C, \mathbb{P}^{d-g})$

smooth of this dim

$$H^1(C, \pi^* T_{\mathbb{P}}) = 0$$

$$H^0(C, \pi^* T_{\mathbb{P}}) = (d-g+1)(d-g+1) + g - 1$$

$$\text{Dimension } \mathrm{Pic}^d C = \dim \mathrm{Mor}(C, \mathbb{P}^{d-g}) - \dim \mathrm{Aut}\, \mathbb{P}^{d-g}$$

$$= (d-g+1)(d-g+1) + g - 1 - \left((d-g+1)^2 - 1\right)$$

$$= g$$

Now let us look more closely at:

$$0 \to T_X \to \pi^* T_Y \to N_{X/Y} \to 0$$

$X \hookrightarrow Y$ both smooth.

The long exact sequence:

$$0 \to \underbrace{H^0(X, T_X)}_{\mathrm{Aut}\, X} \to \boxed{\underbrace{H^0(X, \pi^* T_Y)}_{\mathrm{Def}(X \hookrightarrow Y)}} \to \underbrace{H^0(X, N_{X/Y})}_{\mathrm{Def}(X \hookrightarrow Y)}$$

$$\to \underbrace{H^1(X, T_X)}_{\mathrm{Def}\, X} \to \underbrace{H^1(X, \pi^* T_Y)}_{\mathrm{Ob}(X \hookrightarrow Y)} \to \underbrace{H^1(X, N_{X/Y})}_{\mathrm{Ob}(X \hookrightarrow Y)}$$

$$\to \underbrace{H^2(X, T_X)}_{\mathrm{Ob}\, X} \to H^2(X, \pi^* T_Y) \to H^2(X, N_{X/Y})$$

Aut( )

0 9 4g-3

3g-3 9 0

0 0 0

X_{k°

$\downarrow$

p_{k°

$X \times D \hookrightarrow Y \times D$

X

$D = \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2$