Moduli Spaces in Algebraic Geometry

Math 245 A (winter 2022)

March 2, 2022.

Last time: Let C be projective integral smooth curve / $k = \bar{k}$ of genus g (i.e., $h^1(C, \mathcal{O}_C) = g$).
If $\operatorname{Pic} C$ makes sense, then with some handwaving, we saw that $\operatorname{Pic} C = \coprod_{d \in \mathbb{Z}} \operatorname{Pic}^d C$, and

$$\operatorname{Pic}^d C \cong \operatorname{Pic}^0 C \text{ has dimension } g.$$

First today: with similar handwaving (which we should want to make precise — we can talk about that)

We'll see that "dim $\mathcal{M}_g = 3g-3$." (Question: what if $g \leq 1$?)

We work over $k = \mathbb{C}$. (What will matter will be that char $k = 0$.)

Let $\mathcal{M}_g$ be the moduli space of genus g curves, whatever that means.

Reality check:

What might you hope that it means?

A contravariant FUNCTOR from (k-schemes) to (Sets),

$$B \longmapsto \begin{array}{c} C \\ \downarrow \\ B \end{array}$$

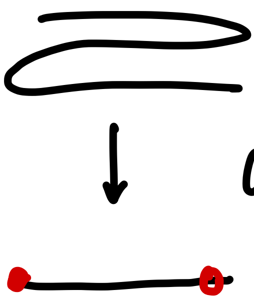
flat proper

geom fibers genus g curves.

smooth connected.

How many degree d genus g covers of $\mathbb{CP}^1$ are there? (Here $d \gg 0$.)

Answer 1: By Riemann-Hurwitz.



genus g

degree d

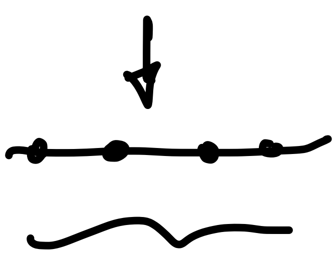
genus 0

$$(2-2g) = d(2-2\cdot 0) - r$$

total ramification

finite # covers

r-dim space



r distinct

$$r = \boxed{2g + 2d - 2}$$

$[s_0, s_1]$ $\quad s_0, s_1 \in \Gamma(C, \mathcal{L}) \quad \deg \mathcal{L} = d$

Answer 2: $\quad C \xrightarrow{[s_0,s_1]} \mathbb{P}^1$

Dimension contribution

choose a curve C $\qquad \dim \mathcal{M}_g$

choose a line bundle $\mathcal{L}$ of degree d. $\qquad + g \quad (\dim \mathrm{Pic}\, C)$

$h^0(\mathcal{L}) = d - g + 1$

choose 2 sections s_0, s_1 $\qquad (d-g+1) + (d-g+1)$.

But they better have no common zeros.

Also, $[s_0, s_1] = [\lambda s_0, \lambda s_1]$; account for scalars $\qquad -1$

$$\dim \mathcal{M}_g + g + 2d - 2g + 2 - 1 = \dim \mathcal{M}_g + 2d - g + 1$$

$$\boxed{2g + \cancel{2d} - 2} = \dim \mathcal{M}_g + \cancel{2d} - g + 1$$

$$\dim \mathcal{M}_g = 3g - 3$$

Construction of Moduli Spaces using our Theorem from last day and last week.

"Moduli space of genus $g>1$ asymmetric curves"

$g=2$ this is the empty set!

Consider the

contravariant FUNCTOR (Schemes) $\to$ (Sets)

$$B \mapsto \begin{array}{c} C \\ \downarrow \\ B \end{array}$$

flat proper smooth

geometric fibers are smooth proper curves, irreducible genus g with no non-identity automorphisms.

Fact (not so easy, I think) $\mathcal{C} \to B$ also has no non-identity auts.

We use the Aut scheme!

$$\begin{tikzcd} & \mathcal{C} \arrow[d] \arrow[r, bend left] & \mathrm{Isom}_B(\mathcal{C},\mathcal{C}) = \mathrm{Aut}_B \mathcal{C} \arrow[dl] \\ B \arrow[r, bend left] & B \arrow[l, bend left, "\mathrm{id}"] \arrow[ur, bend right] & \end{tikzcd}$$

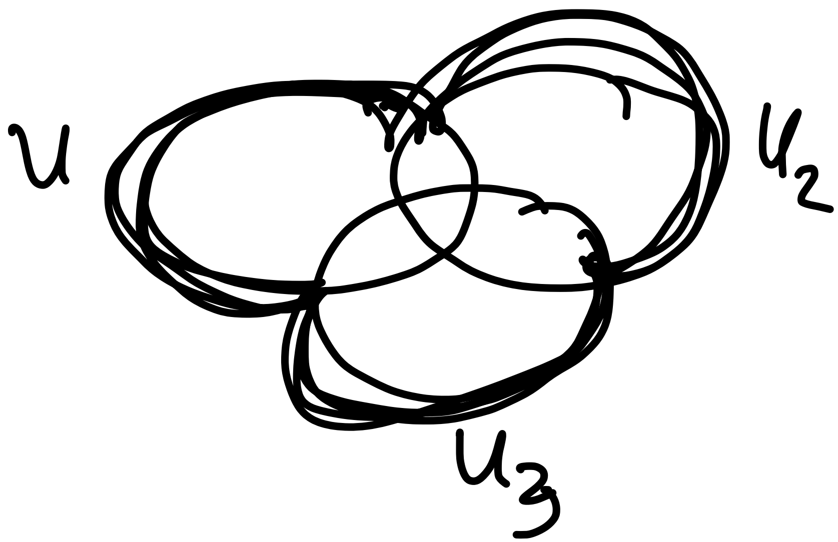
no non-identity automorphism

Any sections?

(Exercise for next day final step)

This will imply
This is a SHEAF.

Worthwhile to write out.



How might we show it is representable?

Fact: by Riemann-Roch + a bit, on any C of genus g (smooth integral over a field),

$$\deg K^{\otimes 2} = 2 \deg K = 4g-4$$

$$h^0(C, K^{\otimes 2}) = \deg(K^{\otimes 2}) - g + 1 = 3g-3$$

and $K^{\otimes 2}$ is very ample.

$$C \xhookrightarrow{H^0(C, K^{\otimes 2})} \mathbb{P}^{3g-2}$$

So we consider the Hilbert scheme of degree $4g-4$ curves in $\mathbb{P}^{3g-2}$. Let's call it Hilb.

It is projective, but parametrizes some crazy stuff.

There is a open locus parametrizing smooth curves.

irreducible.

$C \hookrightarrow \tilde{\mathbb{P}}$

Hilb''

→ $\mathcal{O}(1)\otimes K^{-2} \cong \mathcal{O}$ locally closed. embedded by $K^{\otimes 2}$

$\boxed{\text{Aut scheme}}$ → with no nontrivial automorphisms

Also nondegenerate - Doesn't lie in a hyperplane.

Call this Hilb'

As a FUNCTOR / SHEAF (representable)

Proposition $\quad \mathrm{Hilb}' \longrightarrow \mathcal{M}_g^a$ is a PGL-bundle. $\rightsquigarrow$?

(What does this mean? Why is it true?)

$\square \to B$

PGL-bundle

C

(Exercise)

PGL bundle $\to B$

$\pi_* K_C^{\otimes 2}$

~~(# ($\pi_* K_C^{\otimes 2}$))~~

Upshot: To show $\mathcal{M}_g^a$ is representable, we need to be able to take quotients by PGL (or something like it — GL or SL).

Mumford: invent Geometric Invariant Theory
(variations on this)

Another solution: invent Algebraic Spaces and (Artin?) and declare victory.

Defining $\mathcal{M}_g$ in general: This is not a SHEAF, it is a STACK (not a hard notion despite what people say — but adjacent to some hard notions).

Then $\mathcal{M}_g$ is the STACK/homotopy quotient Hilb''/PGL

suitably defined

Let us now define $\mathrm{Pic}^d C$

as usual, smooth integral curve over a field $k = \bar{k}$

Where am I using this?

Allow me to take $d \gg 0$.

d is so big that for any line bundle of degree d on C, (i) $h^1(C, \mathcal{L}) = 0$ (so $h^0(C, \mathcal{L}) = d - g + 1$)

$d > 2g-2$ suffices for this

(ii) $\mathcal{L}$ is very ample ($d \geq 2g+1$ suffices for this)

$h^0(C,\mathcal{L}) = d-g+1$ $\qquad$ $C \xrightarrow{|\mathcal{L}|} \mathbb{P}^{d-g}$

Then we consider the Hilbert scheme parametrizing curves in $\mathbb{P}^{d-g}$ of degree d and genus g. Call it Hilb, with universal family $\mathcal{U}$.

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[d] & \mathbb{P}^{d-g}\times \mathrm{Hilb} \arrow[dl] \\ \mathrm{Hilb} & \end{tikzcd}$$

$\mathcal{U} \leftrightarrows C\times T$ sm $\hookrightarrow 1$

Consider $\boxed{\mathrm{Isom}_{\mathrm{Hilb}}(\mathcal{U}, C\times \mathrm{Hilb}) / \cancel{PGL(d-g+1)}\ \mathrm{Aut}(\mathbb{P}^{d-g})}$

Toss out those lying in a hyperplane

$\boxed{\text{What is it?}}$ $\quad \mathrm{Pic}^d C$

$C \hookrightarrow \mathbb{P}^{d-g}$ by complete linear series.

$C, \mathcal{L}$ degree d.

$$\left(C \xrightarrow{H^0(C,\mathcal{L})} \mathbb{P}^{d-g} \right) \in \mathrm{Hilb}$$

$$C \times \mathrm{Isom} \longrightarrow \mathbb{P}^{d-g} \times \mathrm{Isom}.$$

$$\underbrace{\mathrm{Pic}^d_C \cong \left(\mathrm{Isom}_{\mathrm{Hilb}}(\mathcal{U}, C \times \mathrm{Hilb}) \setminus \text{degenerate} \right) / \cancel{PGL(d-g+1)}\ \mathrm{Aut}(\mathbb{P}^{d-g})}_{P^{d,g}}$$

degenerate = {C lie in hyperplane}

$$P^{d,g} \to M_g$$

$C, \mathcal{L}$, basis $H^0(C,\mathcal{L})$ / scalar $\longrightarrow$ point of $\left(\mathrm{Isom}_{\mathrm{Hilb}}(\mathcal{U}, C \times \mathrm{Hilb}) \setminus \text{degenerate} \right)$

degenerate = {C lie in hyperplane}