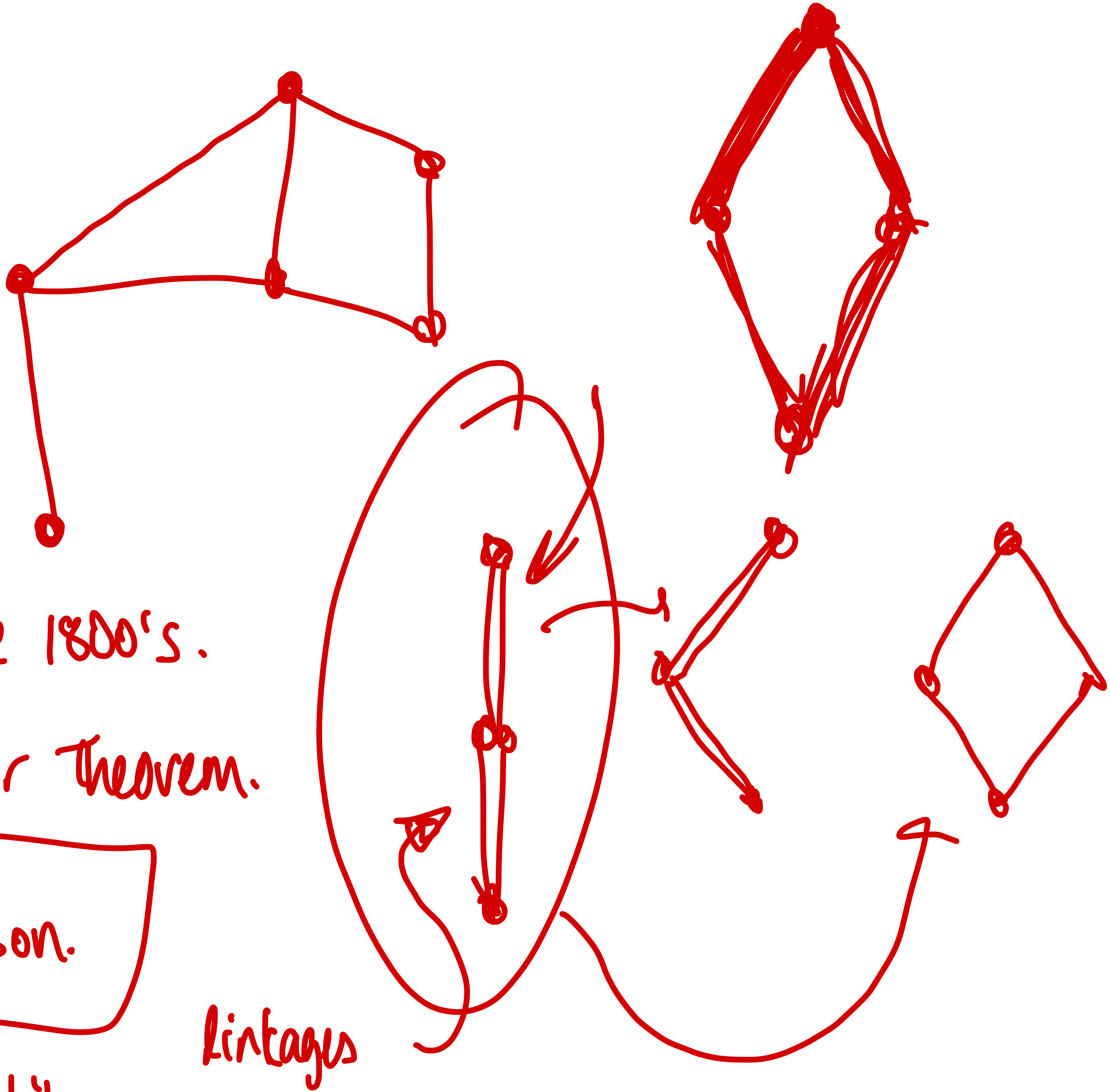
Moduli Spaces in Algebraic Geometry

Math 245A (Winter 2022)

Feb. 28, 2022.



Kempe late 1800's.

Four Color Theorem.

M. J.
Kapovich-Millson.

"universality"

Example: a construction whose configuration space is nonreduced.

Situation:

$$\begin{tikzcd}
\mathcal{L} \arrow[dr, dash] & \\
& X \arrow[d, "\pi"'] \\
& Y
\end{tikzcd}$$

line bundle

"family"

"base of family"

with hypotheses

- π proper
- Y locally Noether. (or: π finitely presented)

+ more to be determined by us

Question: What is the locus $\{q \in Y : \mathcal{L}_q \text{ on } X_q \text{ is trivial}\}$?

Expectation from examples and experience: ...

locally closed subscheme

Theorem

Given:

$$\begin{tikzcd} \mathcal{L} \arrow[r, dash] & X \arrow[d, "\pi"'] \\ & Y \end{tikzcd}$$

proper, flat, geometric fibers are connected and reduced

loc. Noetherian Y

The contravariant **FUNCTOR** $(\text{Schemes}/Y) \to (\text{Sets})$

$$(Z \to Y) \longmapsto \phi_Z : \pi_Z^* \mathcal{M}_Z \xrightarrow{\sim} \mathcal{L}_Z$$

$$\begin{tikzcd} \mathcal{L}_Z \arrow[r, dash] & X_Z \arrow[r] \arrow[d, "\pi_Z"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \arrow[r, dash] & \mathcal{L} \\ \mathcal{M}_Z \arrow[r, dash] & Z \arrow[r] & Y & \end{tikzcd}$$

is a locally closed subfunctor of Y.

bonus:

If furthermore the fibers of π are integral, then the **FUNCTOR** is a closed subfunctor of Y.

Initial Remarks:

1) This generalizes Mumford's generalized SeeSaw Lemma (Abelian Varieties p. 89)

2) This question is local on Y, so we can assume Y is affine, for example.

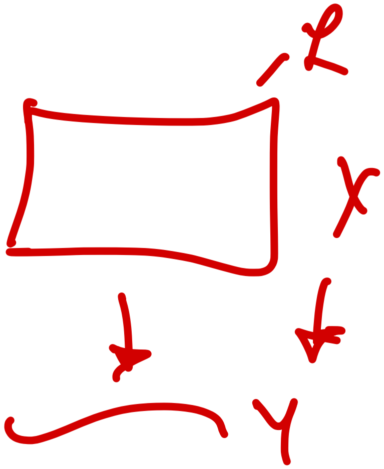
3) As usual, we will first discover/determine/describe $Y_1 \hookrightarrow Y$, and then show it satisfies the universal property. In showing that it satisfies the universal property, it will suffice to check the case where Z is affine as well.

First step (mainly undertaken last ~~day~~ week.):
describe our subset $Y_1 \subset Y$.

Then: explain the scheme structure.

Finally: show the universal property.

Getting the subset $Y_1 \subset Y$



If the fibers are integral:

look at the locus where $h^0(X_q, \mathcal{L}_q) \geq 1$

and $h^0(X_q, \mathcal{L}_q^\vee) \geq 1$.

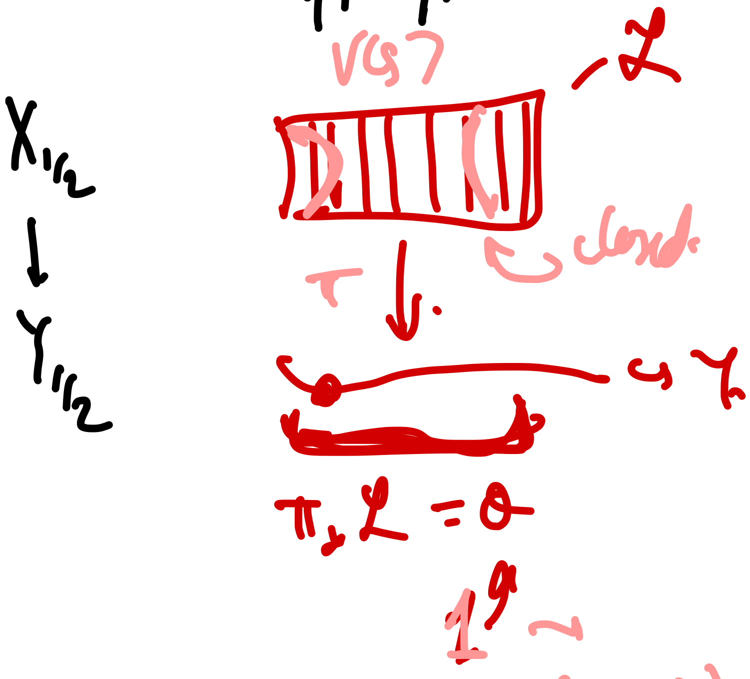
Getting the subset $Y_1 \subset Y$

Without integrality hypotheses:

Consider the locus $Y_{1/2}$ where $h^0(X_q, \mathcal{L}_q) = 1$.

This is locally closed.

Use Grauert here.

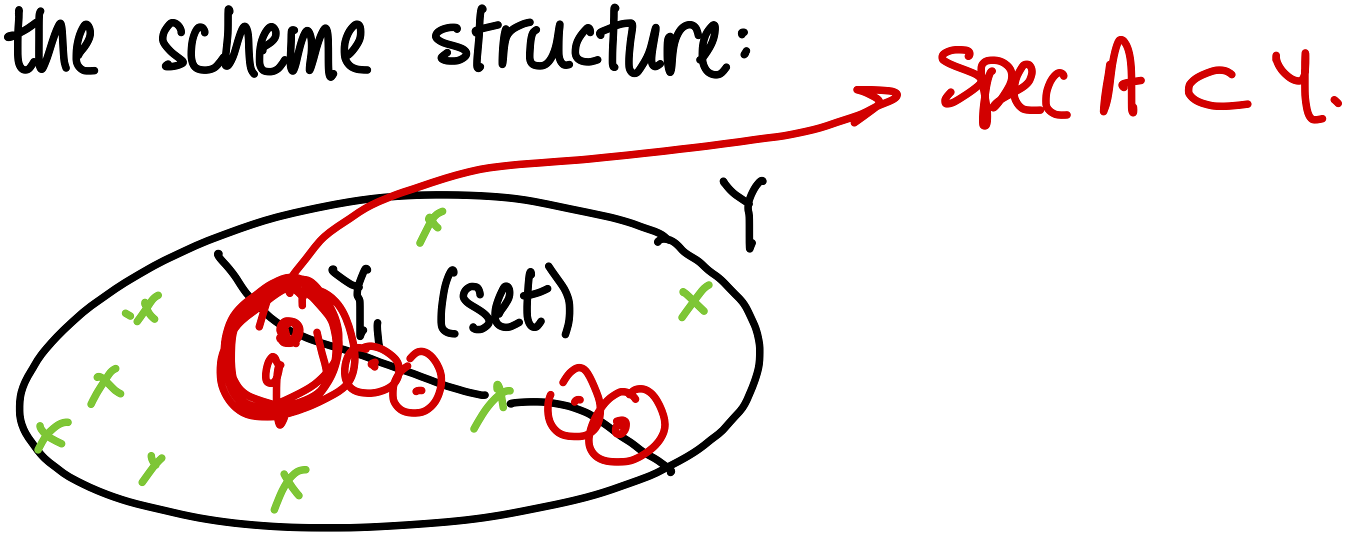


$s \in \Gamma(X_{1/2}, \mathcal{L})$

$\pi_* V(s)$ closed in $Y_{1/2}$.

complement: desired answer.

For the scheme structure:



We describe it in some neighborhood of any given point $q \in Y$.

Discussion...

We need a slight improvement on what we have discussed with the Mumford/Grothendieck/Cohomology-and-Base-Change complex.

situation:

$$\begin{tikzcd} X \arrow[d, "\text{proper}"] \arrow[r, phantom, "\sim"] & \mathcal{F} \text{ coherent; flat}/Y \\ Y = \operatorname{Spec} A & \text{loc. Noetherian} \end{tikzcd}$$

We described:

$$\begin{tikzcd} {} \arrow[r] & A^{s_{-2}} \arrow[r] \arrow[d] & A^{s_{-1}} \arrow[r] \arrow[d] & A^{s_0} \arrow[r] \arrow[d] & A^{s_1} \arrow[r] \arrow[d] & \cdots \arrow[r] & A^{\oplus s_n} \arrow[r] \arrow[d] & A^{\oplus s_n} \arrow[r] \arrow[d] & 0 \\ & 0 \arrow[r] & 0 \arrow[r] & \check{C}^0 \arrow[r] & \check{C}^1 \arrow[r] & \cdots \arrow[r] & \check{C}^{n-1} \arrow[r] & \check{C}^n \arrow[r] & 0 \end{tikzcd}$$

Čech complex for $X, \mathcal{F}$.

This is a quasiisomorphism of complexes

= induces isomorphism of cohomology

= the "total complex" is exact:

$$\cdots \to A^{S_{-1}} \xrightarrow{\sigma_{-1}} A^{S_0} \xrightarrow{\sigma_0} A^{S_1} \oplus \check{C}^0 \to \cdots \to A^{S_n} \oplus \check{C}^{n-1} \to \check{C}^n \to 0$$

so: $0 \to \ker \sigma_0 \to A^{S_1} \oplus \check{C}^0 \to \cdots \to A^{S_n} \oplus \check{C}^{n-1} \to \check{C}^n \to 0$

is exact.

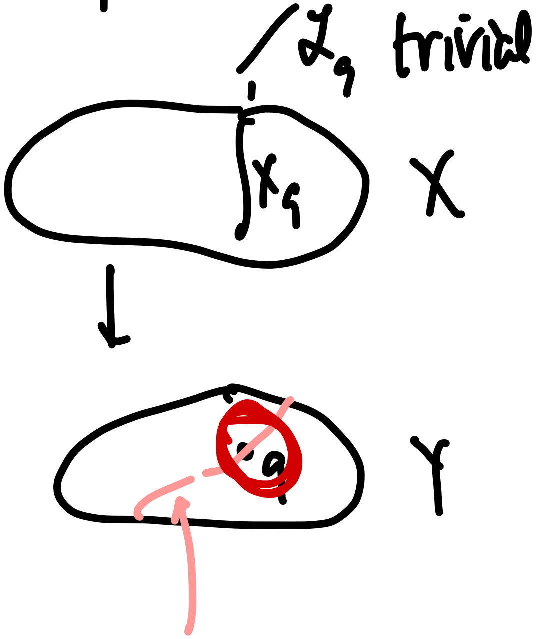
$\ker \sigma_0$ → coherent, flat hence locally free.

$$\begin{tikzcd}
 & & 0 \arrow[r] & \overset{\text{finite rank}}{\text{loc. free}} \arrow[r] \arrow[d] & A^{r_1} \arrow[r] \arrow[d] & \cdots \arrow[r] & A^{r_n} \arrow[r] \arrow[d] & 0 \\
\therefore & & 0 \arrow[r] & \mathcal{C}^0 \arrow[r] & \mathcal{C}^1 \arrow[r] & \cdots \arrow[r] & \mathcal{C}^n \arrow[r] & 0
\end{tikzcd}$$

isomorphism on cohomology, including after <u>any</u> base change A → B.

Back to our proof:

Next, how about the general case? (discuss)



Goal: neighborhood of q in Y.

$q \in \underline{\mathrm{Spec}\, A} \subset Y.$

locus where $\mathcal{L}_q$ is trivial.

$\mathfrak{q} \in \operatorname{Spec} A$

CBC complex for $\mathcal{L}$:

$$0 \to A^{r_0} \xrightarrow{\phi} A^{r_1} \to \cdots \to A^{r_{n-1}} \to A^{r_n} \to 0$$

cohomology here is $H^0(X, \mathcal{L}) = \ker \phi$

Strategy...

$$\begin{tikzcd} & \mathcal{L} \arrow[dl, dash] \\ X \arrow[d] & \\ \operatorname{Spec} A & \end{tikzcd}$$

$$\pi_* \mathcal{L} = \widetilde{H^0(X, \mathcal{L})}$$

Dual as A-modules

TRICK: $A^{r_1} \xrightarrow{\phi^t} A^{r_0} \to M \to 0$ exact

$:= \operatorname{coker} \phi^t$ (A-module)

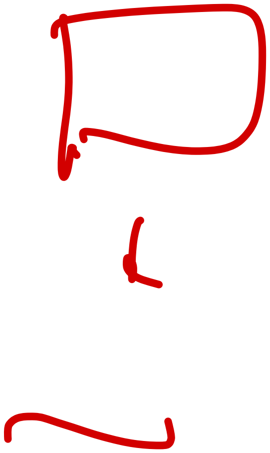
Reason for trick: $B^{r_1} \xrightarrow{\phi_B^t} B^{r_0} \to M \otimes_A B \to 0$ exact

$\Rightarrow 0 \to \operatorname{Hom}_B(M \otimes_A B, B) \to B^{r_0} \xrightarrow{\phi_B} B^{r_1}$ exact

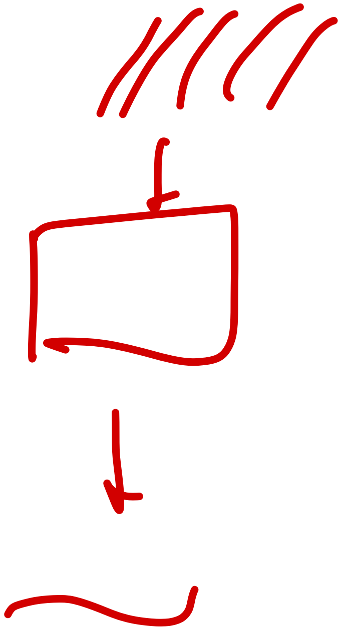
$$\therefore H^0(X_B, \mathcal{L}_B) = \operatorname{Hom}_B(M \otimes_A B, B) = \operatorname{Hom}_A(M, B).$$

From last slide:

$$H^0(X_B, \mathcal{L}_B) = \mathrm{Hom}_B(M \otimes_A B, B) = \mathrm{Hom}_A(M, B).$$



$$\begin{tikzcd}
\mathcal{L}_B \arrow[d, dash] & \mathcal{L} \arrow[d, dash] \\
X_B \arrow[r] \arrow[d, "\pi_B"] & X \arrow[d, "\pi"] \\
\mathrm{Spec}\, B \arrow[r] & \mathrm{Spec}\, A
\end{tikzcd}$$



$$\pi_* \mathcal{L}_B = \widetilde{\mathrm{Hom}_A(M, B)}$$

$\mathcal{O}_q = \mathcal{L}_q$

X_q

q

one section.

Special case: $B = A/\mathfrak{m}$ $[\mathfrak{m}] = q \in \operatorname{Spec} A$

We get:

$$H^0(X_B, \mathcal{L}_B) = \operatorname{Hom}_B(M \otimes_A B, B) = \operatorname{Hom}_A(M, \underset{A/\mathfrak{m}.}{B}).$$

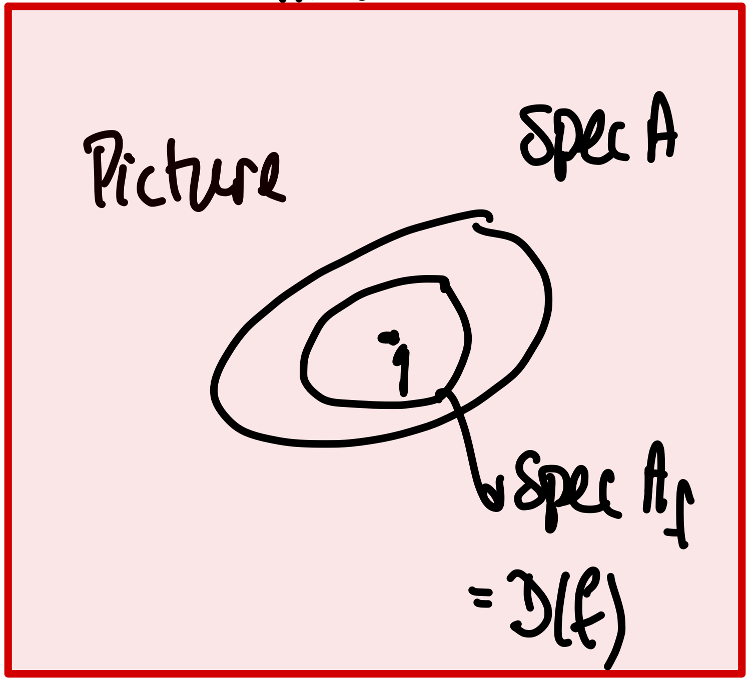
$$\underset{\text{1-dim } \kappa(q)}{} = \operatorname{Hom}_{A/\mathfrak{m}}(\underset{\substack{\uparrow \\ \text{one-dim.}}}{M/\mathfrak{m}M}, \underset{\kappa(q).}{A/\mathfrak{m}}.)$$

$\therefore$ $M/\mathfrak{m}M = \kappa(q)$. Geometric Nakayama:

after shrinking Spec A further,

$\therefore \quad M/_{\mathfrak{m}M} = \kappa(q).$

Geometric Nakayama:



after shrinking Spec A further,

M is generated (as an A-module) by one element.

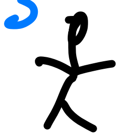
(Replace A by A_f.)

$\therefore \; M \cong \underbrace{A/_I}_{}$ for some I.

Take $B = A/J$.

$$H^0(X_B, \mathcal{L}_B) = \mathrm{Hom}_B(M \otimes_A B, B) = \mathrm{Hom}_A(M, \overset{A/J}{B}).$$

ooo

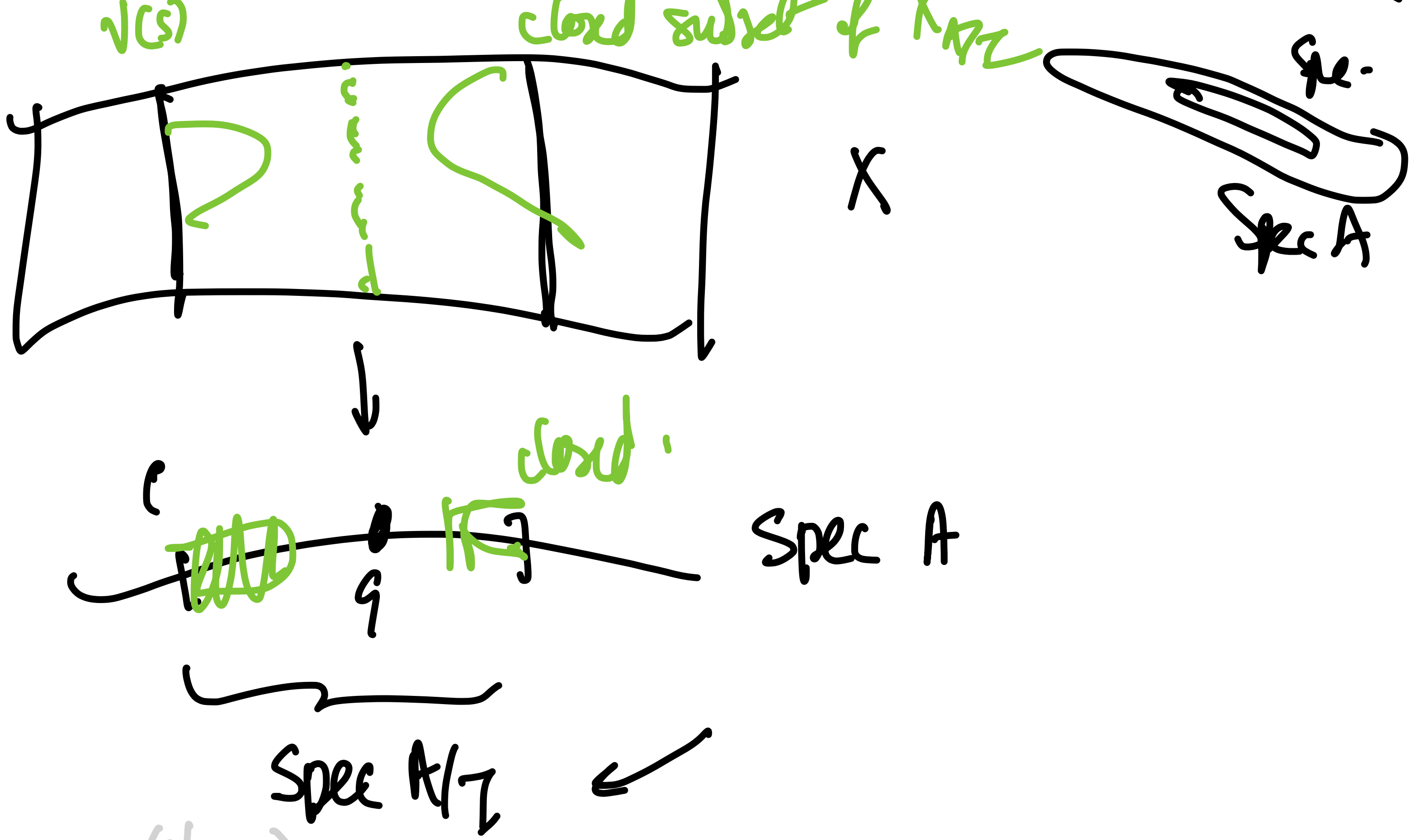


$$\mathrm{Hom}_A(A/I, A/J) = 0$$

$1 \longmapsto$ unless $J \supseteq I$

$1 \to 1$ section s of $\mathcal{L}_{A/I}$.

Picture:



We have a section.

(above)

Conclusion: on an open neighborhood of q in Spec A/I, $\mathcal{L}$ is trivial.

Let's verify the universal property here.

(Shrink A further.)

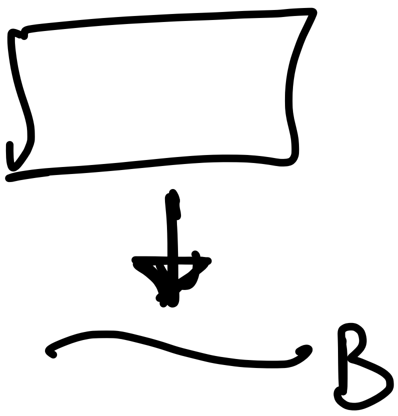
$$\begin{tikzcd}
{} & \mathcal{L}_T = \pi^*\mathcal{M} & {} & \mathcal{L}_{A/I} \cong \mathcal{O} & \mathcal{L} \\
{} & X_T \arrow[u, dash] \arrow[d, dash] & {} & X_{A/I} \arrow[u, dash] \arrow[r] \arrow[d] & X \arrow[u, dash] \arrow[d] \\
\mathcal{M} \arrow[r, dash] & T = \operatorname{Spec} B \arrow[rr, "\text{why??}"] \arrow[rrr, bend right] & {} & \operatorname{Spec} A/I \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

//

Let's use this!

Moduli of smooth automorphism-free curves of genus g

FUNCTOR. B $\mapsto$



flat

geom fibers smooth integral curves of genus g

no nontrivial auts.

isomorphisms

Schemes $\longrightarrow$ Sets

This FUNCTOR is a SHEAF.