Moduli Spaces in Algebraic Geometry

Math 245 A (winter 2022)

Feb. 23, 2022.

Main Plan for the rest of the quarter:

Important tools for
studying / creating / showing the existence of moduli spaces:

- currently: mixing in line bundles
- then (probably): some deformation theory

Our focus, as always, is on how to build robust, adaptable, and versatile tools.

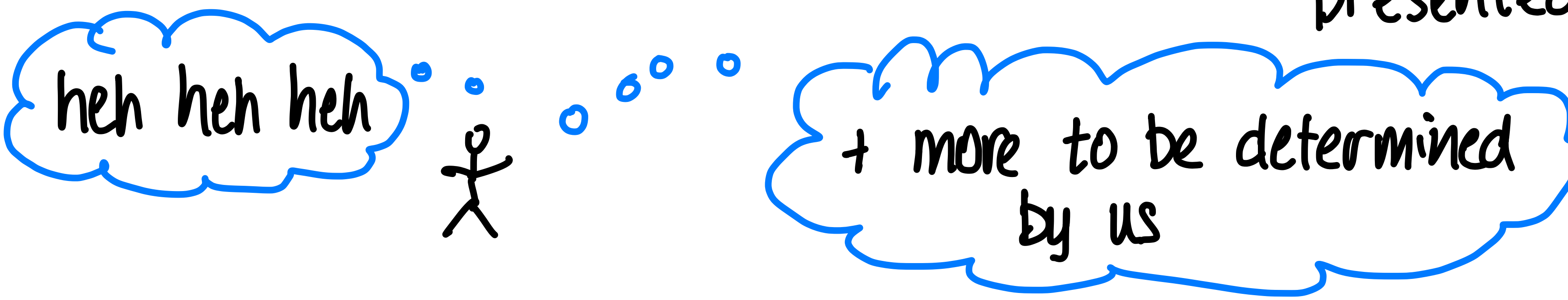
Situation:

$$\begin{tikzcd} \mathcal{L} \arrow[dr, dash] & \\ & X \arrow[d, "\pi"'] \\ & Y \end{tikzcd}$$

line bundle
"family"
"base of family"

with hypotheses

- π proper
- Y locally Noether. (or: π finitely presented)



Question: What is the locus $\{q \in Y : \mathcal{L}_q \text{ on } X_q \text{ is trivial}\}$?

Expectation from examples and experience: ...

locally closed

(Is the answer a subset? Subscheme?)

Motivation:

Suppose X is very nice — say a projective integral curve $C/\mathbb{C}$. Maybe there is a moduli space of line bundles on C, called $\mathrm{Pic}\, C$.* A group scheme! The identity "$\mathcal{O}$" $\in \mathrm{Pic}\, C$ is a closed subscheme.

Then if you had a family $\begin{array}{c} C\times Y \overset{\mathcal{L}}{} \\ \downarrow \\ Y \end{array}$ you'd expect $Y \xrightarrow{\rho} \mathrm{Pic}\, C$.

In this situation, we expect the answer to be

$$\rho^* \mathcal{O} : \quad \mathcal{O} \times_{\mathrm{Pic}\, C} Y \xhookrightarrow{\text{closed}} Y.$$

$$\mathcal{P}ic\, C \longrightarrow \mathrm{Pic}\, C$$

---

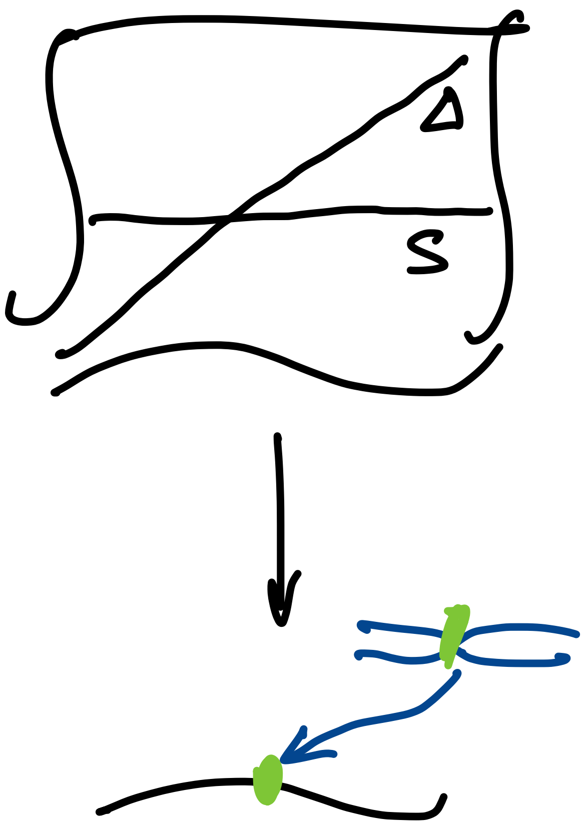
* no such moduli space. There is a moduli stack.

Example where locus is closed:

genus $C > 0$.

X $\quad$ $C \times C$

$\mathcal{L} := \mathcal{O}(\Delta - S)$

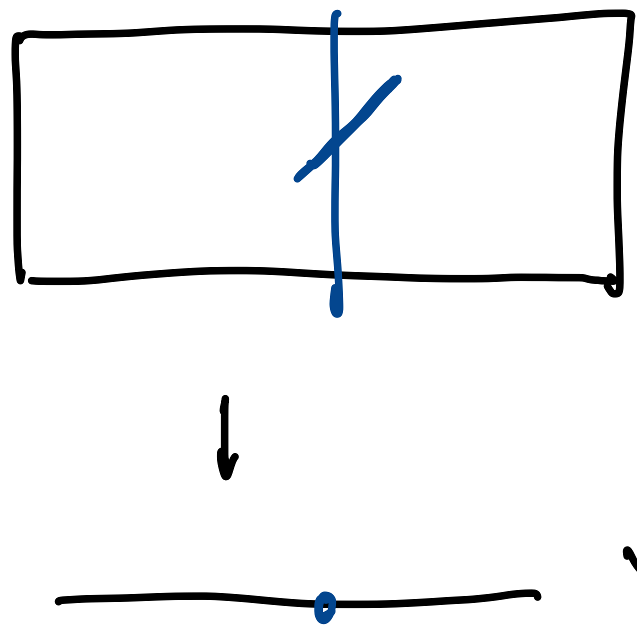


Y $\quad$ C

Example where the locus is nonreduced?

Example where locus is open:

$\mathcal{L} = \mathcal{O}_X(E)$



$X = \mathrm{Bl}\ \mathbb{P}^1 \times Y$

$Y = \mathbb{A}^1$

What is the statement we seek?

$$\begin{tikzcd} \pi^*\mathcal{M} \arrow[r, "\sim"] \arrow[dr, dash] & \mathcal{L} \arrow[d, dash] \\ & X \arrow[d, "\pi"] \\ \mathcal{M} \arrow[r, dash] & Y \end{tikzcd}$$

better than "$\mathcal{L}$ trivial on fibers" is:

"we have a line bundle $\mathcal{M}$ on Y and an isomorphism $\pi^*\mathcal{M} \xrightarrow[\sim]{\phi} \mathcal{L}$.

What is the statement we seek?

Why is it a functor?

Given: $\mathcal{L}$ on X, $\pi : X \to Y$ proper, loc. Noetherian + hypotheses

Consider diagrams.

$\phi_Z : \pi_Z^* \mathcal{M}_Z \xrightarrow{\sim} \mathcal{L}_Z$, $\mathcal{L}_Z$ on X_Z, $\mathcal{M}_Z$ on Z, $\mathcal{L}$ on X

$$\begin{tikzcd} & X_Z \arrow[r] \arrow[d, "\pi_Z"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\pi"] \\ Z' \arrow[r, "j"] & Z \arrow[r] & Y \end{tikzcd}$$

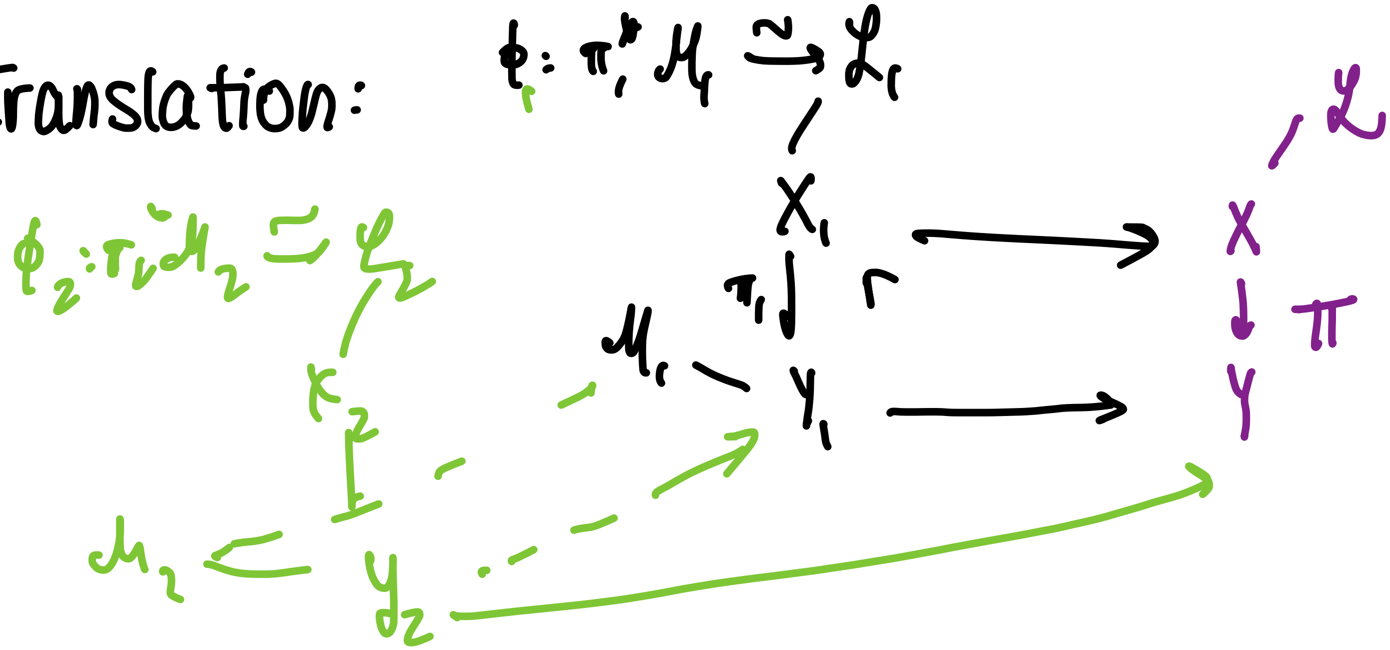
Interpret this as a contravariant FUNCTOR from $(\text{Schemes}/Y)$ to (Sets). (Discuss.)

$(Z \to Y) \longmapsto$ set of such diagrams

Theorem (with missing hypotheses) → $\mathcal{O}$-connected?

This FUNCTOR is representable by some Y-scheme $Y_1 \to Y$.

Translation:



Moreover, $Y_1 \to Y$ is a locally closed embedding.

??

Fancy-sounding restatement: this FUNCTOR is a locally closed subfunctor of Y.

Reasons to like this restatement:

There is no additional content.

It is useful in other situations (e.g. if Y is a stack) that we won't discuss now.

It sounds complicated so you can impress your friend(s).

It is easier to prove.

Things we've shown.

Why would $\mathcal{M}_1$ be determined by Y_1?

$$\begin{array}{ccc} X & - & \pi^*\mathcal{M} \\ {\scriptstyle \pi}\downarrow & & \\ Y & - & \mathcal{M} \end{array}$$

$$\mathcal{O}_{Y_1} \xrightarrow{\sim} \pi_{1*}\mathcal{O}_{X_1}$$

Proposition

Suppose $\pi_1 : X_1 \to Y_1$ is proper and $\mathcal{O}$-connected.

If $\mathcal{M}_1$ is a line bundle (invertible sheaf) on Y_1

then $\mathcal{M}_1 \to \pi_{1*}\pi_1^*\mathcal{M}_1$ is an isomorphism.

Proof: Vaughn.

Remark: If Y is reduced, then triviality of $\mathcal{L}$ on fibers implies $\mathcal{L}$ is a pullback in good circumstances.

Proposition

Situation:

$$\begin{tikzcd} & \mathcal{L} \arrow[dl, dash] \\ X \arrow[d, "\pi"] \\ Y \end{tikzcd}$$

$\mathcal{L}$ invertible sheaf

π flat, proper, $\mathcal{O}$-connected

Y locally Noetherian

If Y is reduced, and for all $q \in Y$, $\mathcal{L}_q$ is trivial on the fiber X_q, then

(a) $\mathcal{M} := \pi_* \mathcal{L}$ is an invertible sheaf on Y, and

(b) $\pi^* \mathcal{M} \longrightarrow \mathcal{L}$ is an isomorphism.

More on $\mathcal{O}$-connectedness:

Done earlier: **Proposition.** If W is a proper $\bar{k}$-scheme that is ~~geometrically~~ connected and ~~geometrically~~ reduced, then $H^0(W, \mathcal{O}_W) = \bar{k}$: the only functions on the variety are constants.

Reminder: if $\alpha \in H^0(W, \mathcal{O}_W)$ $\quad W \to \mathbb{A}^1 \subset \mathbb{P}^1$

Important Exercise If $\pi: X \to Y$ is proper, flat, Y is locally Noetherian, and the fibers of π satisfy $h^0(X_q, \mathcal{O}_{X_q}) = 1$. Then π is $\mathcal{O}$-connected (i.e., $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$ is an isomorphism). (Pf uses cohomology and Base Change Theorem, see The Rising Sea.)

Theorem

Given: loc. Noetherian

$$\begin{array}{c} \mathcal{L} \\ X \\ \pi \downarrow \\ Y \end{array}$$

proper, flat, geometric fibers are connected and reduced

(maybe even integral, but not yet)

The contravariant FUNCTOR (Schemes$/Y$) $\to$ (Sets)

$(Z \to Y) \longmapsto$ $\phi_Z : \pi_Z^* \mathcal{M}_Z \xrightarrow{\sim} \mathcal{L}_Z$

$$\begin{tikzcd} \mathcal{L}_Z \arrow[dash, r] & X_Z \arrow[r] \arrow[d, "\pi_Z"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\pi"] \arrow[dash, r] & \mathcal{L} \\ \mathcal{M}_Z \arrow[dash, r] & Z \arrow[r] & Y & \end{tikzcd}$$

is a locally closed subfunctor of Y.

Theorem

Given: $\mathcal{L} \frown X \xrightarrow{\pi} Y$, π proper, flat, geometric fibers are connected and reduced (maybe even integral, but not yet), Y loc. Noetherian.

The contravariant FUNCTOR $(\text{Schemes}/Y) \to (\text{Set})$

$$(Z \to Y) \longmapsto \left\{ \phi_Z : \pi_Z^* \mathcal{M}_Z \xrightarrow{\sim} \mathcal{L}_Z \right\}$$

$$\begin{tikzcd} \mathcal{L}_Z \frown X_Z \arrow[r] \arrow[d, "\pi_Z"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \frown \mathcal{L} \arrow[d, "\pi"] \\ \mathcal{M}_Z \frown Z \arrow[r] & Y \end{tikzcd}$$

is a locally closed subfunctor of Y.

Initial Remarks:

1) This generalizes Mumford's generalized SeeSaw Lemma (Abelian Varieties p. 89)

2) This question is local on Y, so we can assume Y is affine, for example.

Theorem

Given: $\begin{array}{c} \mathcal{L} \smallsetminus \\ X \\ \pi \downarrow \\ Y \end{array}$ proper, flat, geometric fibers are connected and reduced (maybe even integral, but not yet)

loc. Noetherian Y

The contravariant FUNCTOR $(\text{Schemes}/Y) \to (\text{Sets})$

$$(Z \to Y) \longmapsto \begin{tikzcd} & X_Z \arrow[r] \arrow[d, "\pi_Z"] & X \arrow[d, "\pi"] \\ \mathcal{M}_Z & Z \arrow[r] & Y \end{tikzcd} \quad \phi_Z : \pi_Z^* \mathcal{M}_Z \xrightarrow{\sim} \mathcal{L}_Z$$

is a locally closed subfunctor of Y.

3) As usual, we will first discover/determine/describe $Y_1 \hookrightarrow Y$, and then show it satisfies the universal property. In showing that it satisfies the universal property, it will suffice to check the case where Z is affine as well.

How can you tell if a line bundle is trivial?

Proposition: If X/k is (not) integral and proper, and $\mathcal{L}$ is a line bundle on X, and $h^0(X,\mathcal{L})>0$ and $h^0(X,\mathcal{L}^\vee)>0$, then $\mathcal{L}\cong\mathcal{O}$. (and conversely!)

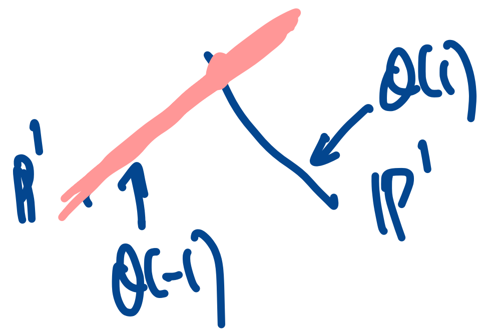
Proof: $0\neq s\in H^0(X,\mathcal{L})$
$0\neq t\in H^0(X,\mathcal{L}^\vee)$.

$0\neq st\in H^0(X,\mathcal{O})$

$\therefore$ st is a constant, $\neq 0$

s is nowhere zero.

$\boxed{\mathcal{O}\xrightarrow{\times s}\mathcal{L} \text{ is an iso.}}$



$\boxed{\begin{array}{l} h^0(\mathcal{L})=1 \\ h^0(\mathcal{L}^\vee)=1.\end{array}}$

Question:

If X/k is merely proper, geometrically connected, and geometrically reduced, what can we say?

set-theoretic version:

Theorem

Given:
$$\begin{tikzcd} \mathcal{L} \arrow[r, dash] & X \arrow[d, "\pi"'] \\ & Y \end{tikzcd}$$
proper, flat, geometric fibers are connected and reduced ~~(maybe even~~ $\boxed{\text{integral}}$ ~~but not yet)~~

loc. Noetherian $Y^{red} = Y$

The contravariant FUNCTOR $\left(\overset{\text{reduced}}{\text{Schemes}}/Y\right) \to (\text{Sets})$

$$(Z \to Y) \longmapsto \left\{ \phi_Z : \pi_Z^* \mathcal{M}_Z \xrightarrow{\sim} \mathcal{L}_Z \qquad \begin{tikzcd} \mathcal{L}_Z \arrow[r, dash] & X_Z \arrow[r] \arrow[d, "\pi_Z"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\pi"] \arrow[r, dash] & \mathcal{L} \\ \mathcal{M}_Z \arrow[r, dash] & Z \arrow[r] & Y & \end{tikzcd} \right.$$

is a ~~locally~~ closed subfunctor of Y^{red}.

Let's prove this!

locus where $h^0(X_{y}, \mathcal{L}^\vee) \geq 1$ and $h^0(X_{y_1}, \mathcal{L}) \geq 1$ closed subset

✓ First: What is the closed subset Y_1 ?

✓ Second: Why is

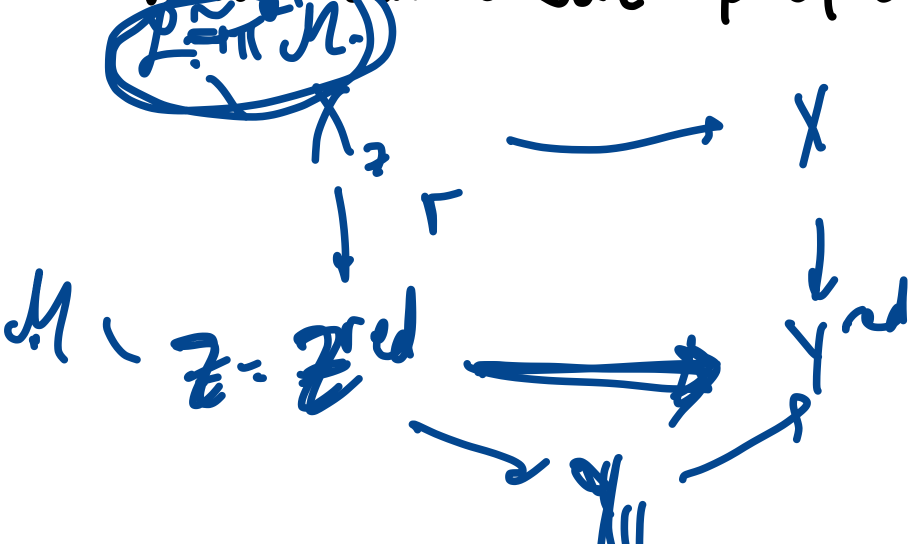
$$\begin{array}{ccc} \mathcal{L}_1 \searrow & & \\ & X_1 & \\ & \downarrow & \\ & Y_1 & \end{array}$$

trivial on fibers

Y_1 reduced

the pullback of some $\mathcal{M}_1 \searrow Y_1$?

Third: Universal property?



set-theoretic version:

Theorem

Given: $\mathcal{L} \searrow X \xrightarrow{\pi} Y$ proper, flat, geometric fibers are connected and reduced ~~(maybe even~~ $\boxed{\text{integral}}$ ~~but not yet)~~

loc. Noetherian $Y^{red} = Y$

The contravariant FUNCTOR $(\text{reduced Schemes}/_Y) \to (\text{Set})$

$$(Z \to Y) \longmapsto \begin{array}{ccccc} \phi_Z : \pi_Z^* \mathcal{M}_Z \xrightarrow{\sim} \mathcal{L}_Z \searrow & X_Z & \longrightarrow & X & \nearrow \mathcal{L} \\ & \downarrow \pi_Z & \ulcorner & \downarrow \pi & \\ \mathcal{M}_Z \searrow & Z & \longrightarrow & Y & \end{array}$$

is a ~~$\boxed{\text{locally}}$~~ closed subfunctor of Y^{red}.

What do you do set-theoretically w/o integral.

$$\begin{array}{ccc} & & \mathcal{L} \\ & \diagup & \\ X & & \\ \downarrow & \text{proper flat; } \overset{\text{geo}}{\text{fibers}} \text{ connected + reduced.} & \\ Y & \text{l. Noeth} & \end{array}$$

tell me why locus of pts $g \in Y$ where $\mathcal{L}|_{X_g}$ is trivial

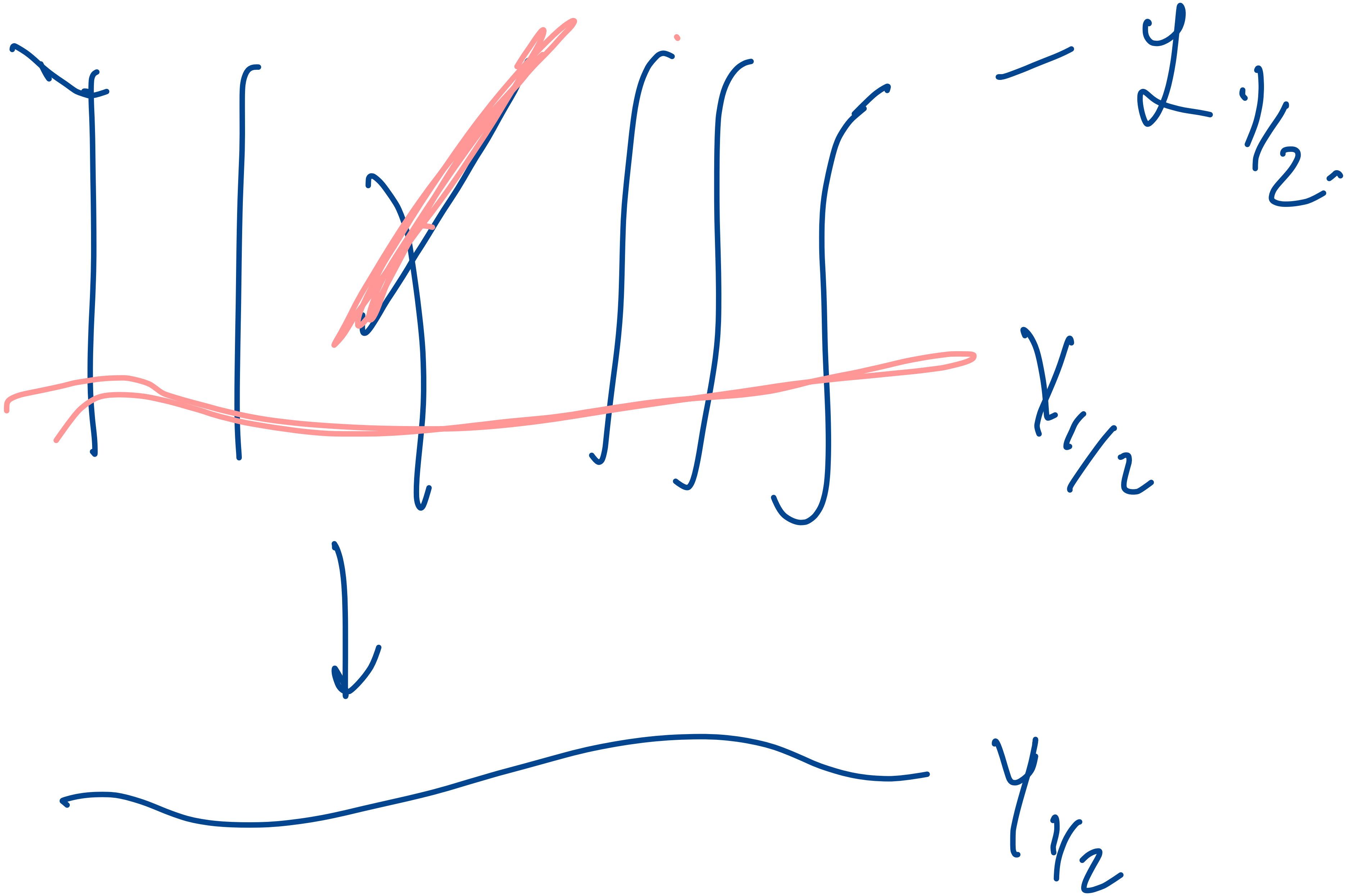
loc closed?

Answer: look @ locus where $\boxed{h^0(\mathcal{L}_g) \geq 1}$ $h^0(\mathcal{L}^{-1}_g) \geq 1$ $Y_{1/2}$

locally closed subset. But I may have too much.

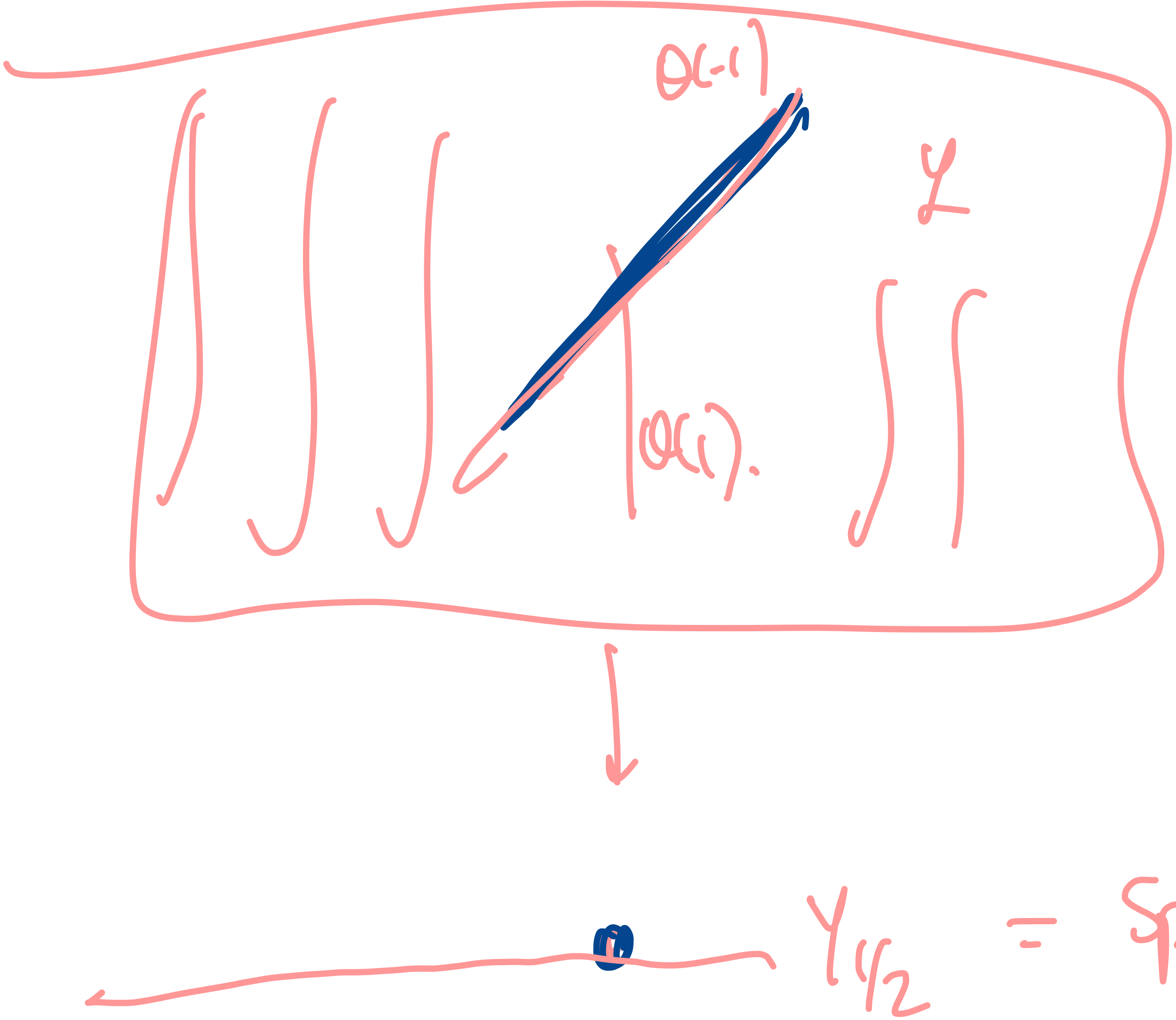
$Y_{\mathcal{L}_2}$ $\mathcal{L}_2$. $Y_{\mathcal{L}_2}$.

$\pi_* \mathcal{L}_{1/2}$ line bundle by Grauert.



$\pi_* \mathcal{L}$ is a line bundle.

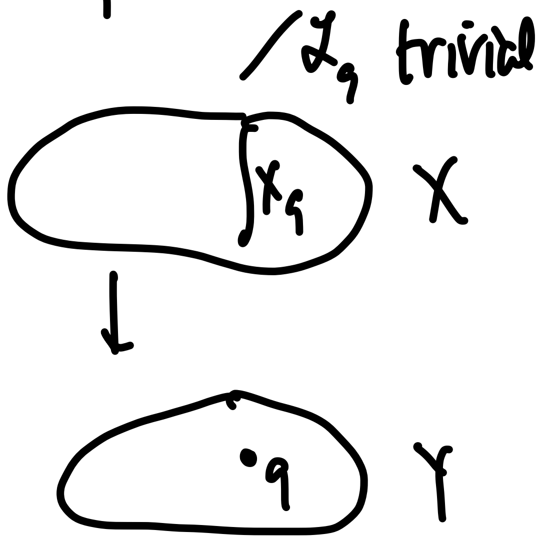
$= \mathcal{O}$.



$h^0(Y) = 1.$

$Y_{1/2} = \operatorname{Spec} k.$

Next, how about the general case? (discuss)



Goal: neighborhood of q in Y.

$q \in \operatorname{Spec} A \subset Y$.

Cohomology and Base Change Complex:

careful!

$$0 \to A^0 \to A^1 \to \cdots \to A^{r_n - 1} \to A^{r_n} \to 0$$

cohomology here is $H^0(X, \mathcal{L})$

Strategy...

$$\pi_* \mathcal{L} = \widetilde{H^0(X, \mathcal{L})}$$

define

$$A^{r_1} \xrightarrow{\phi^t} A^{r_0} \longrightarrow M \longrightarrow 0$$

For any A-algebra B,

$$B^{r_1} \xrightarrow{\phi^t} B^{r_0} \longrightarrow M \otimes_A B \longrightarrow 0 \quad \text{is exact}$$

$$0 \longrightarrow \mathrm{Hom}_B(M \otimes_A B, B) \longrightarrow B^{r_0} \xrightarrow{\phi} B^{r_1} \quad \text{is exact}$$

$$= \mathrm{Hom}_A(M, B)$$

Thus:

$$\begin{tikzcd} \mathcal{L}_B \arrow[d, dash, dashed] & \mathcal{L} \arrow[d, dash, dashed] \\ X_B \arrow[r] \arrow[d, "\pi_B"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

$$\pi_{B*} \mathcal{L}_B = \overbrace{H^0(X_B, \mathcal{L}_B)}$$

$$= \mathrm{Hom}_A(M, B)$$ ← new!