Moduli Spaces in Algebraic Geometry

Math 245 A (Winter 2022)

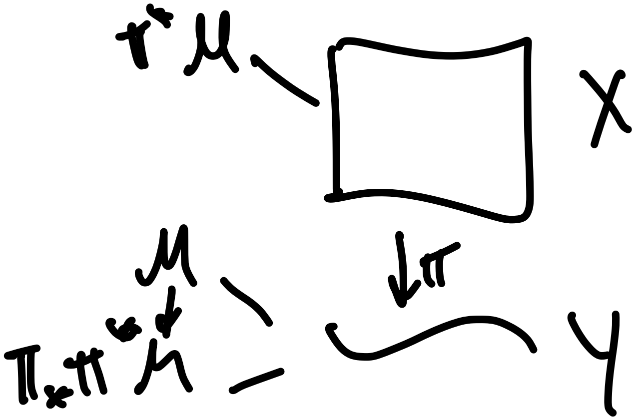
Feb. 21, 2022.

$\mathcal{O}_Y \xrightarrow{\sim} \pi_* \mathcal{O}_X$

Proposition

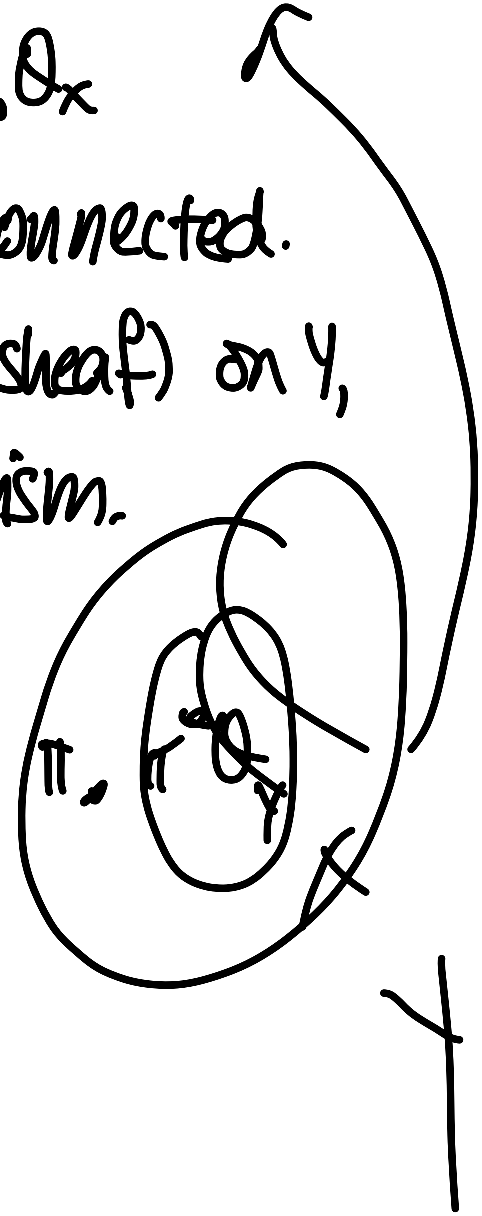
Suppose $\pi : X \to Y$ is proper and $\mathcal{O}$-connected.
If $\mathcal{M}$ is a line bundle (invertible sheaf) on Y,
then $\boxed{\mathcal{M} \to \pi_* \pi^* \mathcal{M}}$ is an isomorphism.

Proof:



If $\mathcal{M}$ is $\mathcal{O}$:

$\mathcal{O}_Y \xrightarrow{\sim} \pi_* \pi^* \mathcal{O}_Y$



Proposition

Situation: $\mathcal{L}$ invertible sheaf on X, $\pi: X \to Y$ flat, proper, geo. conn. + geo. red. fibers ($\Rightarrow$ universally $\mathcal{O}$-connected), Y locally Noetherian.

If Y is reduced, and for all $q \in Y$, $\mathcal{L}_q$ is trivial on X_q (the fiber), then (a) $\mathcal{M} := \pi_* \mathcal{L}$ is an invertible sheaf on Y, and (b) $\pi^* \mathcal{M} \to \mathcal{L}$ is an isomorphism.

(Do we need reduced hypotheses?) → Yes! Exercise!

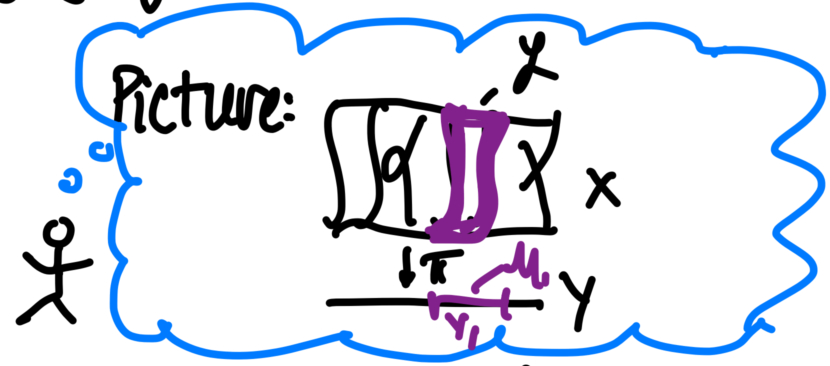
Proof: (a) Grauert

Always!: $\pi^* \pi_* \mathcal{L} \to \mathcal{L}$

(b) Suppose $\mathcal{M} \cong \mathcal{O}$. Want: $\pi^* \mathcal{M} \to \mathcal{L}$ is an iso at all points. (l.b.)

The next result is the focus of this week. On Wednesday, I will start the discussion of this from the beginning.

Proposition (generalizing Mumford's generalized SeeSaw Lemma, Abelian Varieties p. 89)



Situation: $X \xrightarrow{\pi} Y$, $\mathcal{L}$ invertible sheaf on X

π flat, proper, geo. conn. + geo. red. fibers

($\Rightarrow$ universally 0-connected)

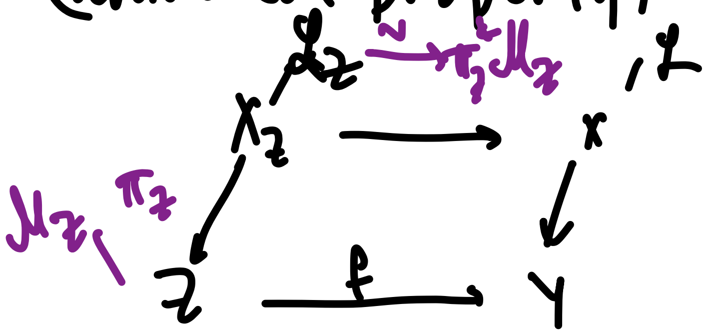
integral

Y locally Noetherian

There is a "best" closed subscheme $Y_1 \hookrightarrow Y$ and a line bundle $\mathcal{M}_1$ on Y_1 and an isomorphism $\pi^* \mathcal{M}_1 \xrightarrow{\sim} \mathcal{L}$ on $X_1 := X \times_Y Y_1$, such that:

(universal property) For all $f: Z \to Y$ such that...

with $\mathcal{M}_Z = f^* \mathcal{M}_1$

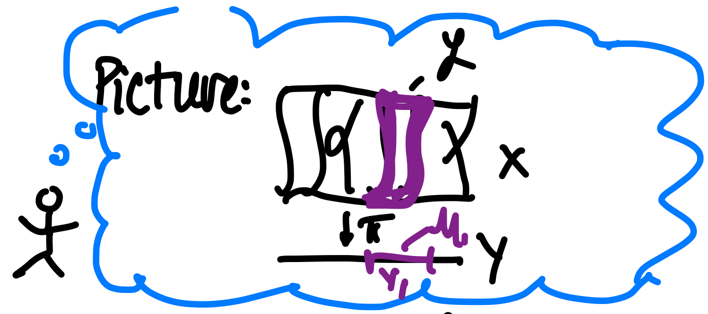


iff $Z \xrightarrow{f} Y$ factors through Y_1.

Proof: soon.

Version for closed subsets.

Proposition



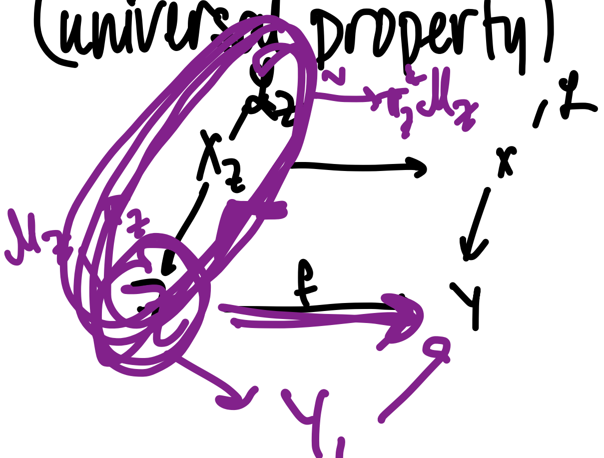
Situation: $\mathcal{L}$ invertible sheaf on X

$X \xrightarrow{\pi} Y$, π flat, proper, geo. conn. + geo. red. fibers ($\Rightarrow$ universally $\mathcal{O}$-connected)

Y locally Noetherian reduced

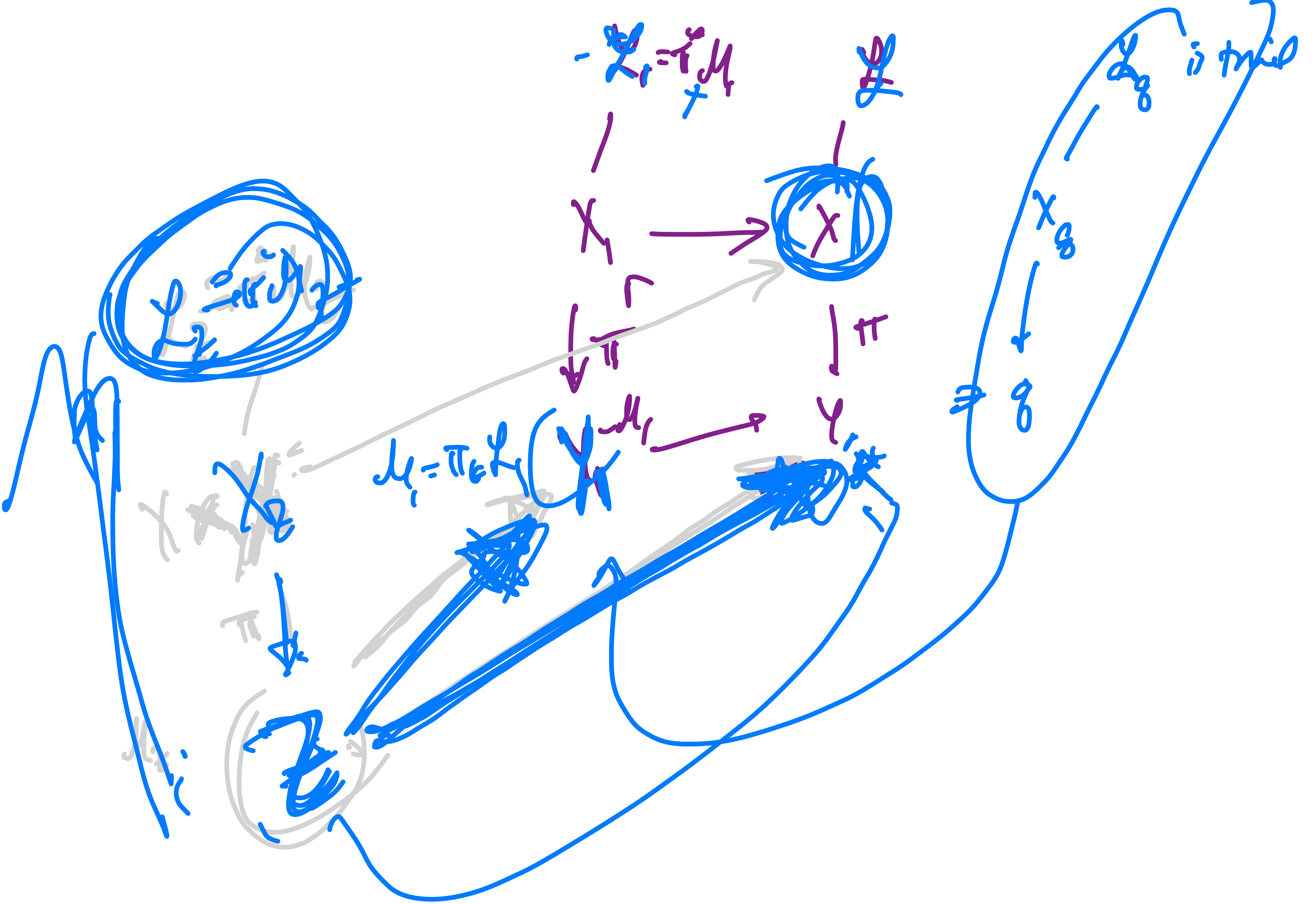
There is a "best" closed subset reduced closed subscheme $Y_1 \hookrightarrow Y$ and a line bundle $\mathcal{M}_1$ on Y_1 and an isomorphism $\pi^*\mathcal{M}_1 \xrightarrow{\sim} \mathcal{L}$ on $X_1 := X \times_Y Y_1$, such that:

(universal property) For all reduced $f: Z \to Y$ such that...



with $\mathcal{M}_Z = f^*\mathcal{M}_1$

iff f factors through Y_1: $Z \xrightarrow{f} Y_1$.



X_1
X
Y_1
Y
X_2
Z
X_q
q
π
$\mathcal{L}_q$ is trivial
$\mathcal{M}_1 = \pi_* \mathcal{L}_1$

Lemma Suppose $X \to Y$ is proper with geo. connected and reduced fibers.

$$\begin{array}{c} \mathcal{L} \\ | \\ X \\ \downarrow \\ Y \end{array}$$

$\mathcal{L}$ is a line bundle on X.

Then the locus of points q on Y where $\mathcal{L}_q$ is trivial on X_q is a closed subset. (This implies the "set-theoretic statement on the previous slide.).

Proof. For $\mathcal{L}_q$ on a geo. connected + reduced proper variety, $\mathcal{L}_q \cong \mathcal{O}$ if and only if $h^0(\mathcal{L}_q) \geq 1$ and $h^0(\mathcal{L}_q^\vee) \geq 1$.

By Semicontinuity Thm, $h^0(\mathcal{L}_q) : Y \to \mathbb{Z}^{\geq 0}$ is usc, $h^0(\mathcal{L}_q^\vee)$ is too, so our locus is closed.

soon

IOU: Why is it that if we have a nonzero section s of $\mathcal{L}$ on a variety X and a nonzero section t of $\mathcal{L}^\vee$ on X, why is $\mathcal{L} \xrightarrow{\sim} \mathcal{O}$?

Hypotheses: X irreducible.

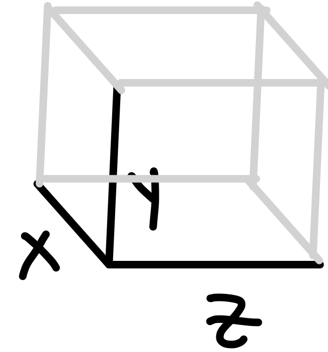
st is a nonzero section of $\mathcal{O}$

But $h^0(X, \mathcal{O}) = 1$ (geom. integral) proper.

st is a nonzero constant

So s is a nowhere vanishing section of $\mathcal{L}$. So $\mathcal{L}$ is trivial. $\mathcal{O} \xrightarrow{\times s} \mathcal{L}$. //

Application: Theorem of the Cube



Suppose X and Y are proper integral k-varieties, Z connected k-scheme, $\mathcal{L}$ a line bundle on $X \times Y \times Z$ whose restrictions to

$$\{x_0\} \times Y \times Z, \quad X \times \{y_0\} \times Z, \quad X \times Y \times \{z_0\} \text{ are trivial}$$

k-points of X, Y, Z respectively.

Then $\mathcal{L}$ is trivial.

Picture:

Proof: soon.

Proof:

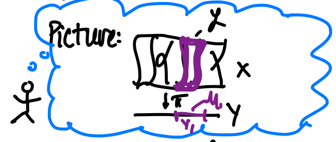
Uniqueness: ... ✓

Next, Y_1 determines $\mathcal{M}_1$: ✓

$$\mathcal{M}_1 = \pi_* \mathcal{L}|_{X_1} \quad \ldots$$

$$\begin{array}{ccc} X_1 & - & \mathcal{L}_1 = \pi^* \mathcal{M}_1 \\ \downarrow & & \\ Y_1 & - & \pi_* \mathcal{L}_1 = \mathcal{M}_1 \end{array}$$

Proposition (generalizing Mumford's generalized SeeSaw Lemma, Abelian Varieties p. 89)



Situation: $\pi: X \to Y$ with $\mathcal{L}$ an invertible sheaf on X; π flat, proper, geo. conn. + geo. red. fibers ($\Rightarrow$ universally $\mathcal{O}$-connected); Y locally Noetherian.

There is a "best" ✓ closed subscheme $Y_1 \hookrightarrow Y$ and a line bundle $\mathcal{M}_1$ on Y_1 and an isomorphism $\pi^*\mathcal{M}_1 \xrightarrow{\sim} \mathcal{L}$ on $X_1 := X \times_Y Y_1$, such that:

(universal property) For all $f: Z \to Y$ such that... (with $\mathcal{M}_Z = f^*\mathcal{M}_1$)

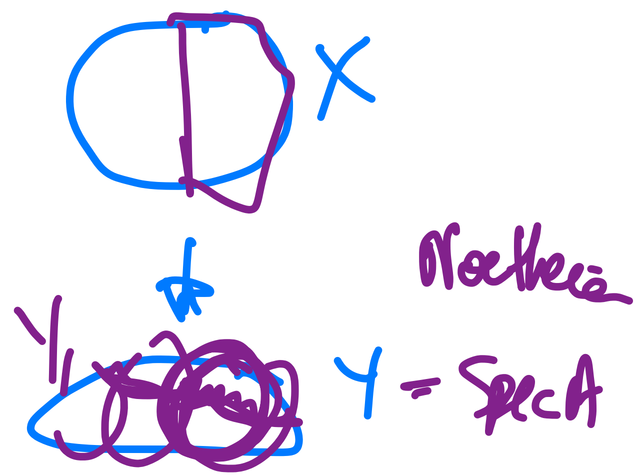
$$\begin{array}{ccc} X_Z & \longrightarrow & X \\ \downarrow \pi_Z & & \downarrow \\ Z & \xrightarrow{f} & Y \end{array}$$

($\mathcal{L}_Z \xrightarrow{\sim} \pi_Z^*\mathcal{M}_Z$, $\mathcal{M}_Z$, $\mathcal{L}$)

iff $Z \xrightarrow{f} Y$ factors through Y_1.

Proof: soon.

Reduce to the case where Y is affine, $Y = \operatorname{Spec} A$. ...



Cohomology + Base Change

$X \ni \mathcal{L}$, $X \to \operatorname{Spec} A$

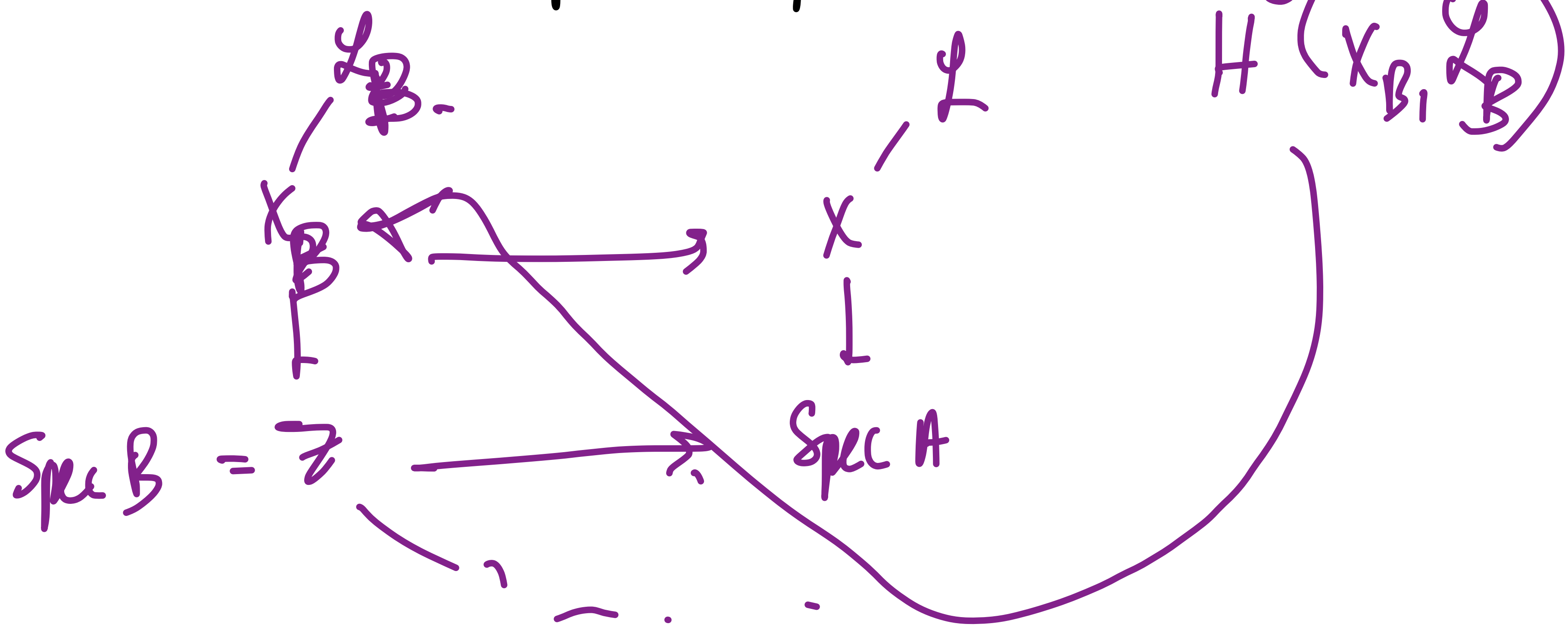
(Grothendieck-Mumford) complex:

careful!

$$0 \to A^{r_0} \xrightarrow{\phi} A^{r_1} \to \cdots \to A^{r_n} \to 0$$

cohomology is $H^0(X,\mathcal{L})$ $H^1(X,\mathcal{L})$

We care about $\operatorname{Spec} B \to \operatorname{Spec} A$ ...



$0 \to A^{r_0} \xrightarrow{\phi} A^{r_1} \to \cdots$, $\ker \phi = H^0$

define $A^{r_1} \xrightarrow{\phi^t} A^{r_0} \to M \to 0$

For any A-algebra B,

$$B^{r_1} \xrightarrow{\phi^t} B^{r_0} \to M \otimes_A B \to 0 \quad \text{is exact}$$

$$0 \to \mathrm{Hom}_B(M \otimes_A B, B) \to B^{r_0} \xrightarrow{\phi} B^{r_1} \quad \text{is exact}$$

$$= \mathrm{Hom}_A(M, B)$$

Thus:

$$\begin{array}{ccc} X_B \ \text{-}\text{-}\ \mathcal{L}_B & \longrightarrow & X \ \text{-}\text{-}\ \mathcal{L} \\ \pi_B \downarrow \quad \ulcorner & & \downarrow \pi \\ \mathrm{Spec}\, B & \longrightarrow & \mathrm{Spec}\, A \end{array}$$

$$\pi_{B*}\mathcal{L}_B = \overbrace{H^0(X_B, \mathcal{L}_B)} = \mathrm{Hom}_A(M, B)$$

← new!

We will continue from here.