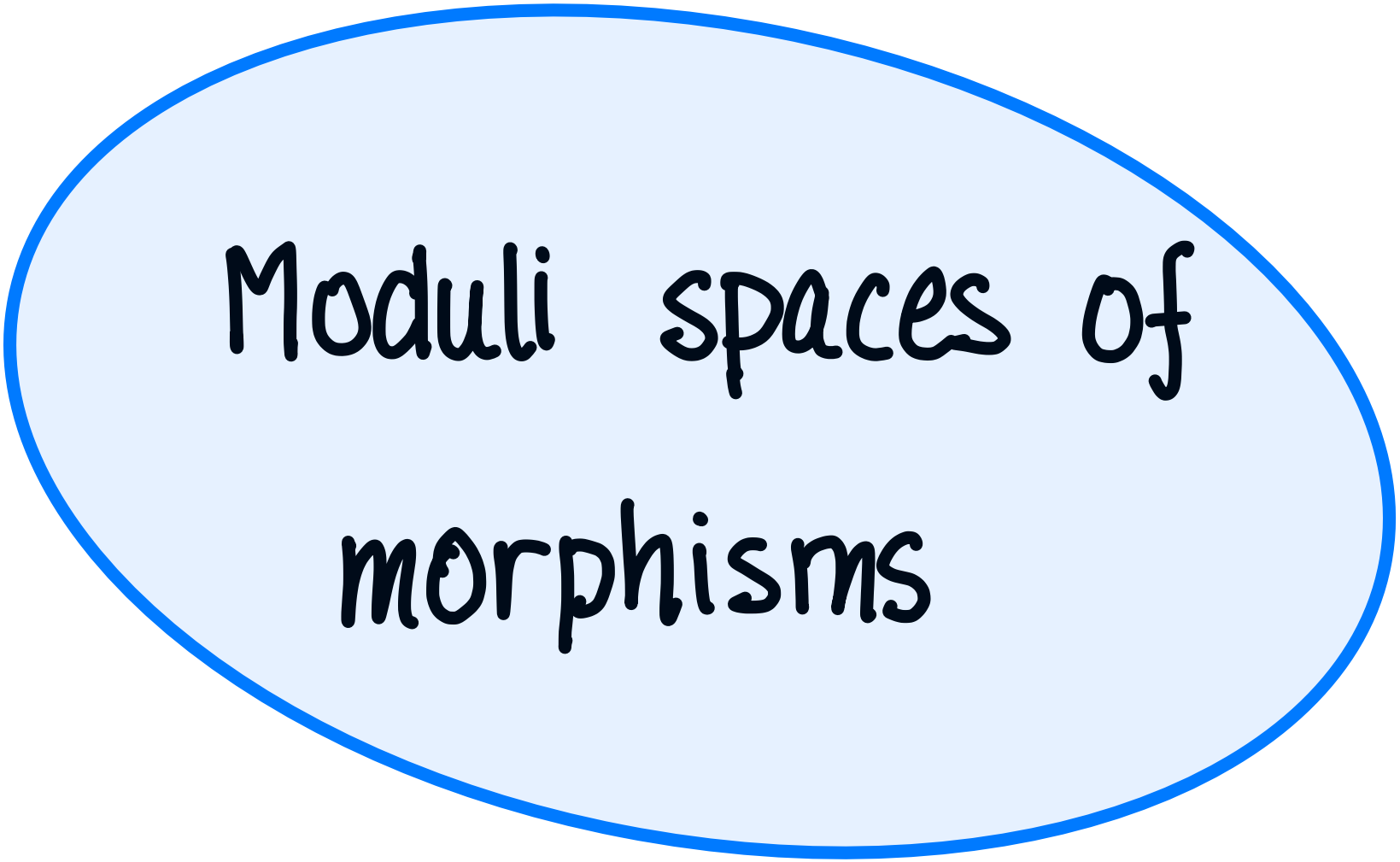
Moduli Spaces in Algebraic Geometry

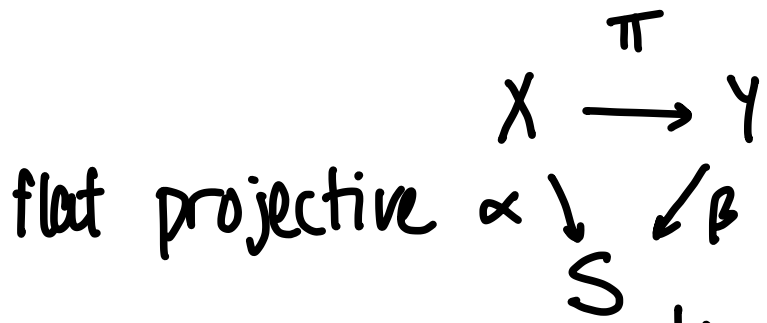
Math 245 A (winter 2022)

Feb. 16, 2022.



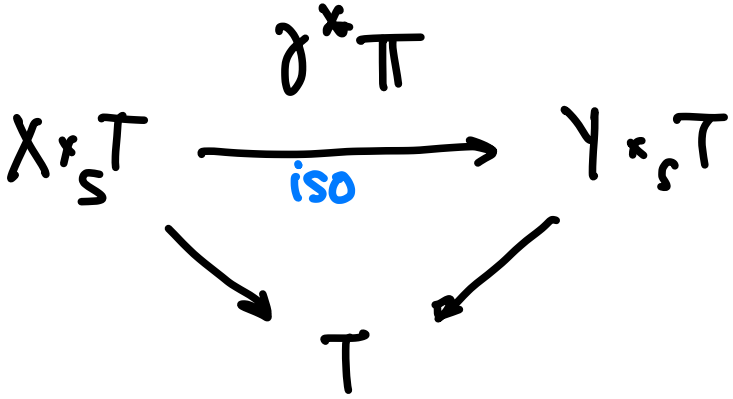
To define them, we need to prove:

Theorem Given



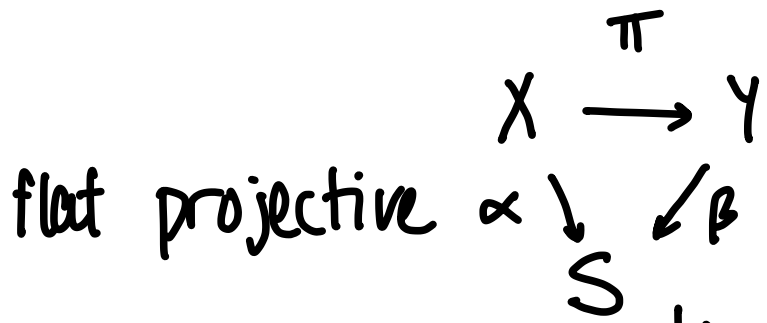
Then there is an open subset (subscheme) $U \subset S$ satisfying the following universal property:

$\gamma : T \to S$ satisfies



if and only if γ factors through $U \xhookrightarrow{\text{open}} S$.

Theorem' Given



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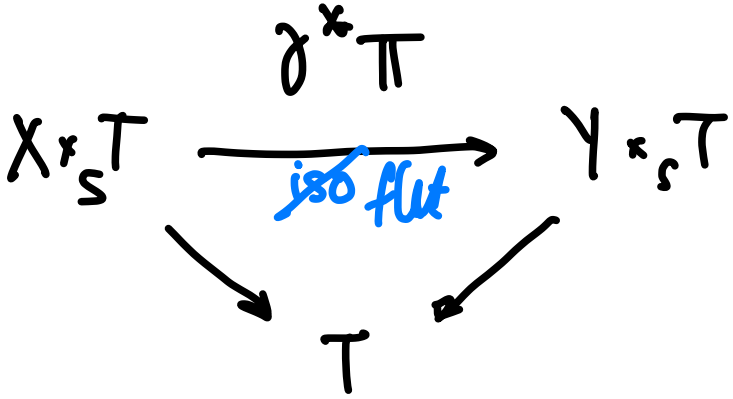


if and only if γ factors through $U \hookrightarrow S$.
open

worth remembering: → HARD!

Fancy Flatness Facts

(not proved in "The Rising Sea", for example)

1) "flat maps are open" in reasonable circumstances

$$X \xrightarrow{\pi} Y$$

flat, finite type
locally Noetherian

π(open) = open

2) "the flat locus is open" in reasonable circumstances

$$X \xrightarrow{\pi} Y$$

finite type
locally noetherian

flat locus in X = open

3) "fibral flatness criterion"

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & S & \end{array}$$

$\sigma\epsilon$

finite type, flat
locally Noetherian

π_σ flat $\Leftrightarrow$ π flat

Theorem Given $X \xrightarrow{\pi} Y$ ($\therefore$ projective), flat projective $\alpha: X \to S$, $\beta: Y \to S$ projective flat, locally Noetherian.

Then there is an open subset (subscheme) $U \subset S$ satisfying the following universal property:

$\gamma: T \to S$ satisfies $X \times_S T \xrightarrow{\gamma^*\pi} Y \times_S T$ (iso, flat), with $X \times_S T \to T$ and $Y \times_S T \to T$,

if and only if γ factors through $U \hookrightarrow S$ (open).

Fancy Flatness Facts
(not proved in "The Rising Sea", for example)

1) "flat maps are open" in reasonable circumstances

$X \xrightarrow{\pi} Y$ flat, finite type, locally Noetherian — $\pi(\text{open}) = \text{open}$

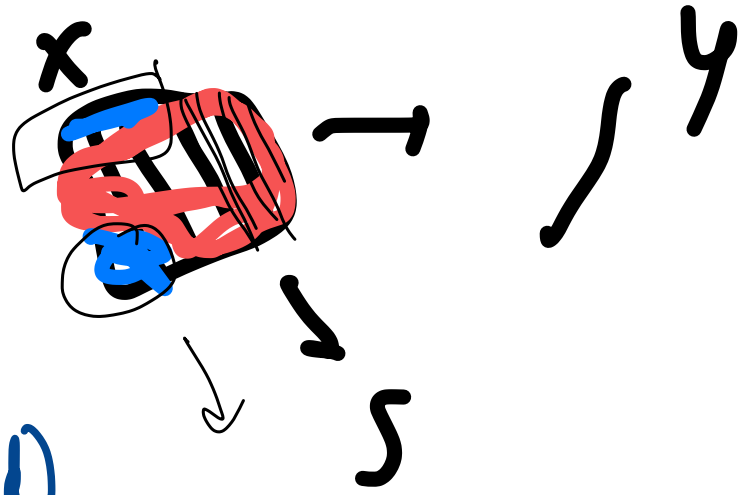
2) "the flat locus is open" in reasonable circumstances

$X \xrightarrow{\pi} Y$ finite type, locally noetherian — flat locus in X = open

3) "fibral flatness criterion"

$X \xrightarrow{\pi} Y$, $\alpha: X \to S$, $\beta: Y \to S$, $\alpha \in S$ — finite type, flat, locally Noetherian — π_α flat $\Leftrightarrow \pi$ flat

Proof As usual, we first find our candidate U, then show that it has the desired universal property.



What is U?

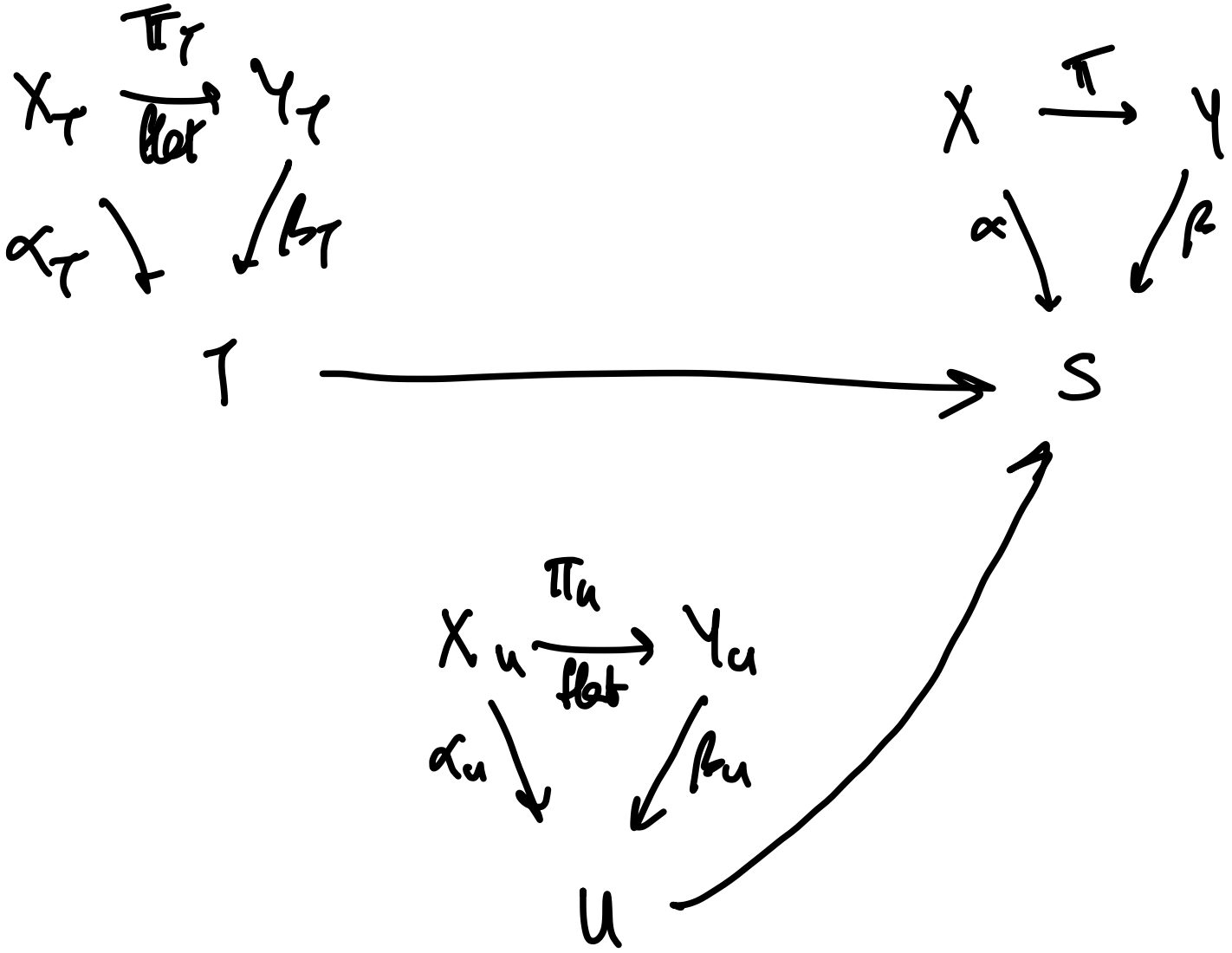
locus in X where π is flat (open)

complement in X (closed)

Apply α. (closed)

Take complement (open) (in S) U.

Why does it satisfy the universal property?



Proposition

Suppose:
- $X \to Y$ morphism of k-schemes
- l/k is a field extension
- $X_l \to Y_l$ flat

Then $X \to Y$ is flat

Proof Reduce to the case where X and Y are affine

$\operatorname{Spec} A \to \operatorname{Spec} B$ i.e. $B \to A$

Is $\cdot \otimes_B A$ exact?

We know $\cdot \otimes_{B_l} A_l$ is exact.

$0 \to M \to N \to P \to 0$

If

$0 \to M \otimes_B A \to N \otimes_B A \to P \otimes_B A \to 0$

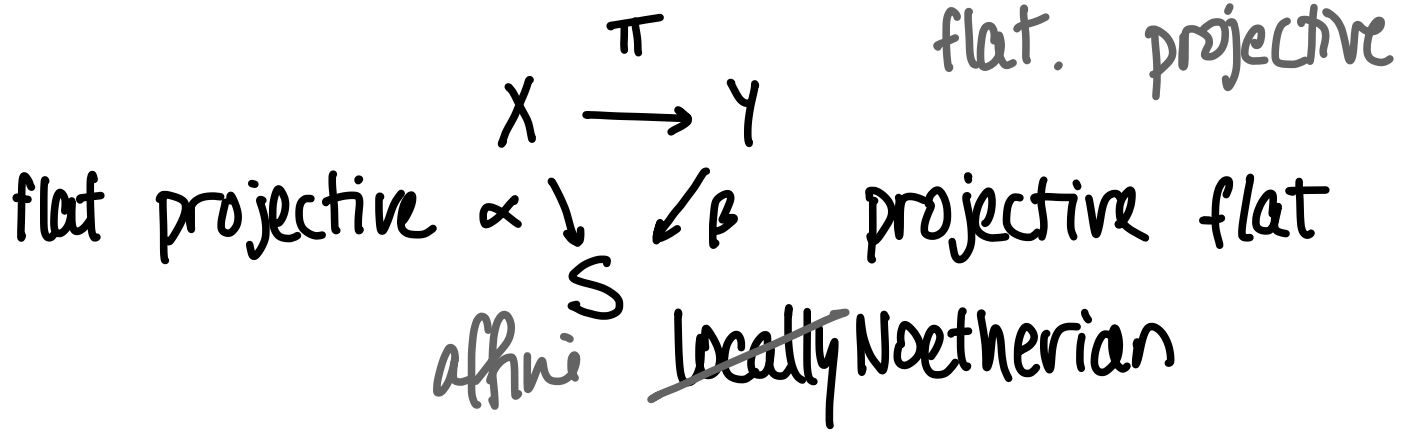
$\otimes_k l$

$0 \to M \otimes_k l \to N \otimes_k l \to P \otimes_k l \to 0$

B-module

Exact?

Theorem Given



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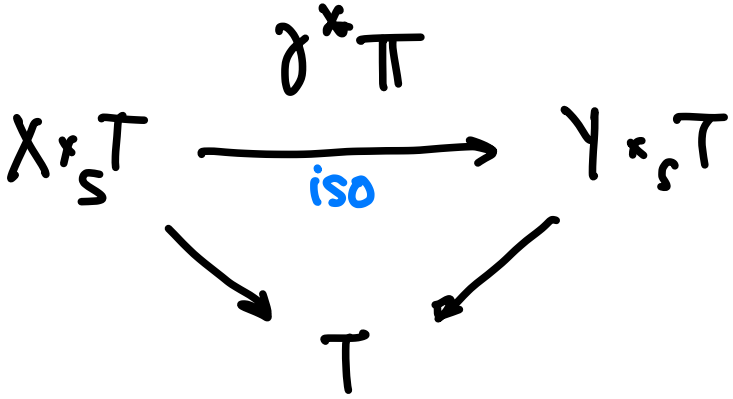
$\gamma: T \to S$ satisfies



if and only if γ factors through $U \xhookrightarrow[\text{open}]{} S$.

Theorem Given $X \xrightarrow{\pi} Y$ (flat, projective)

$\alpha \downarrow$ flat projective, $\swarrow \beta$ projective flat

S (affine, ~~locally~~ Noetherian)

Then there is an open subset (subscheme) $U \subset S$ satisfying the following universal property:

$\gamma: T \to S$ satisfies $X \times_S T \xrightarrow{\gamma^*\pi} Y \times_S T$ (iso), with $X \times_S T \to T \leftarrow Y \times_S T$

if and only if γ factors through $U \hookrightarrow S$ (open).

$X \xrightarrow{\pi} Y$ projective.

By uppersemicontinuity of fiber dimension, there is a closed subset Y' of Y where the fiber dimension of π is > 0.

Toss out $\beta(Y')$ in S

leaving open subset of S.

Theorem Given $X \xrightarrow{\pi} Y$ (flat, projective)

α flat projective, β projective flat, to S

affine, ~~locally~~ Noetherian

Then there is an open subset (subscheme) $U \subset S$ satisfying the following universal property:

$\gamma: T \to S$ satisfies $X \times_S T \xrightarrow[\text{iso}]{\gamma^* \pi} Y \times_S T$, with both to T,

if and only if γ factors through $U \hookrightarrow S$ (open).

fiber dim 0. $\therefore$ finite.

Next, $\pi(X)$ is open;

let $Y'' \subset Y$ be the (closed) complement.

$\beta(Y'') \subset S$ is closed.

Toss it out.

Can now assume π surjective.

Theorem Given $X \xrightarrow{\pi} Y$ (flat, projective)

α: flat projective, β: projective flat, to S

S: affine, ~~locally~~ Noetherian

Then there is an open subset (subscheme) $U \subset S$ satisfying the following universal property:

$\gamma: T \to S$ satisfies $X \times_S T \xrightarrow{\gamma^*\pi}_{\text{iso}} Y \times_S T$, with both maps to T

if and only if γ factors through $U \hookrightarrow S$ (open).

fiber dim 0. $\therefore$ finite. surjective.

Next, $\pi_* \mathcal{O}_X$ is a coherent sheaf on Y.

By uppersemicontinuity of rank of $\pi_* \mathcal{O}_X$, there is a closed subset $Z''' \subset Y$ where the rank is ≥ 2.

Take image in S. Toss it out.

Theorem Given $X \xrightarrow{\pi} Y$ (flat, projective)

X flat projective over S via α, Y projective flat over S via β

affine, ~~locally~~ Noetherian

Then there is an open subset (subscheme) $U \subset S$ satisfying the following universal property:

$\gamma: T \to S$ satisfies $X \times_S T \xrightarrow{\gamma^*\pi}_{\text{iso}} Y \times_S T$, with both maps to T,

if and only if γ factors through $U \hookrightarrow S$ (open).

fiber dim 0. $\therefore$ finite.

surjective.

degree 1 on points.

know:

$\pi_* \mathcal{O}_X = \mathcal{O}_Y$

~~That's what's left~~

But what about:

not deg 1 @ all pts

QED!

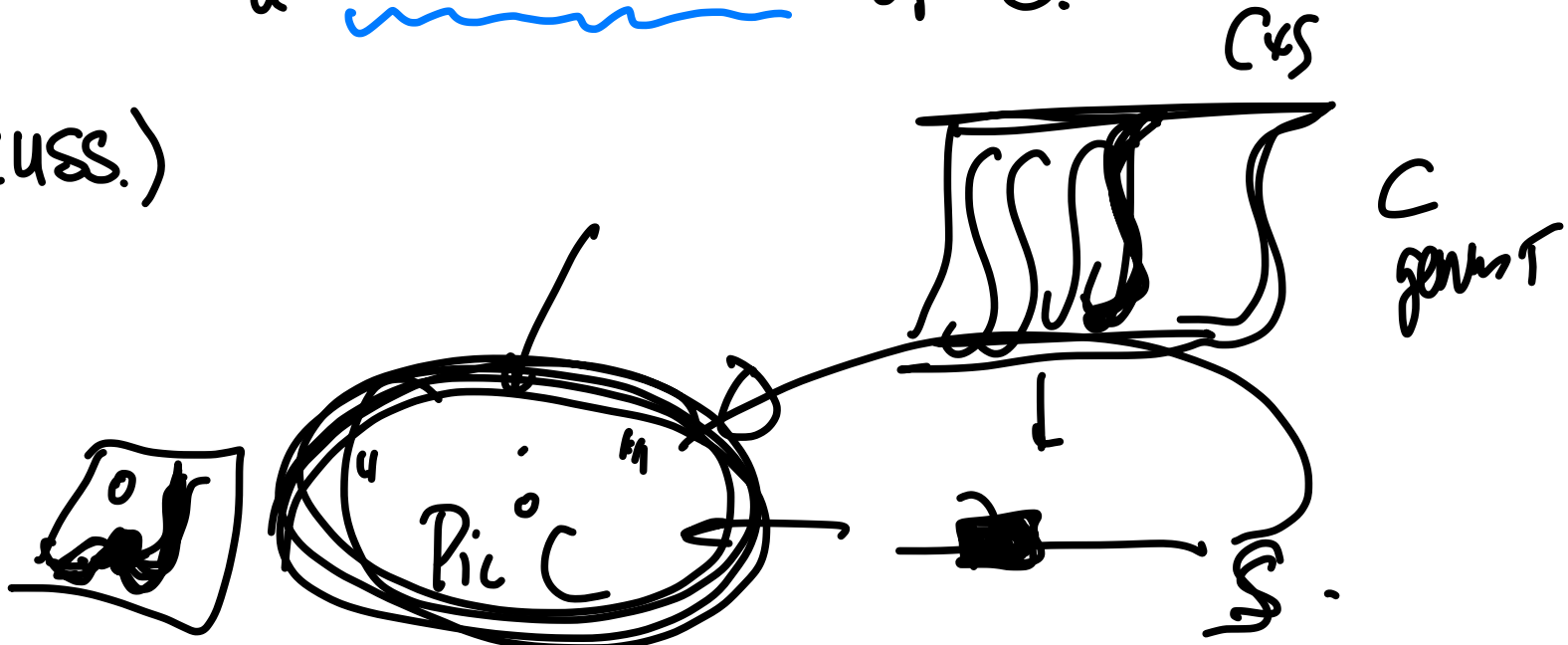
Another closed condition to make new moduli spaces from old

Situation: $\mathcal{L}$ invertible sheaf

$$\begin{array}{c} \mathcal{L} \\ \nearrow \\ X \\ \downarrow \\ S \end{array}$$

Idea: The locus on S where $\mathcal{L}$ is trivial is a ~~~~~ of S.

Lucia Caporaso (Discuss.)

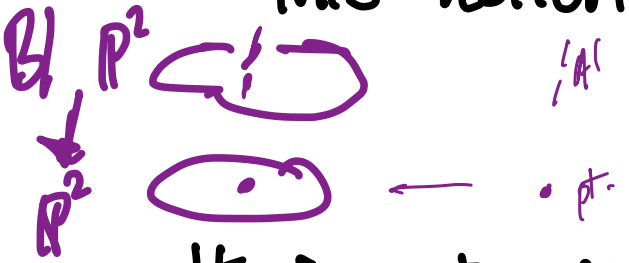


Useful: if further. $R^{i>0}\pi_* \mathcal{O}_X = 0$

Definition

A morphism $\pi: X \to Y$ such that $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$ is an isomorphism is 0-connected.

This notion is clearly preserved by composition and local on the target.

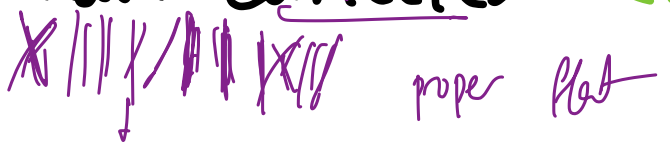


It is not necessarily preserved by base change (example?), thanks! but it is preserved by flat base change. (cohomology/pushforward of quasicoherent sheaves commutes with flat base change: use the FTHF theorem on the Čech complex.)

Basic facts about 0-connected morphisms

Exercise: proper 0-connected morphisms are surjective.

***Theorem (Zariski's Connectedness Lemma):** Proper 0-connected morphisms have connected (nonempty) fibers.



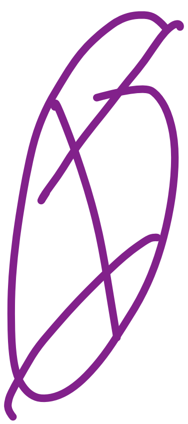
Proposition: Suppose $\pi: X \to Y$ is proper, flat, and (Y reduced) Y is locally Noetherian. Suppose the fibers satisfy $h^0(X_q, \mathcal{O}_{X_q}) = 1$. Then π is 0-connected.

Proof:

Proposition: If X is proper $/\bar{k}$, and ~~geometrically~~ connected and ~~geometrically~~ reduced, then $h^0(X, \mathcal{O}_X) = 1$.

Proof:

base change to $\bar{k}$.



proper over $\bar{k}$

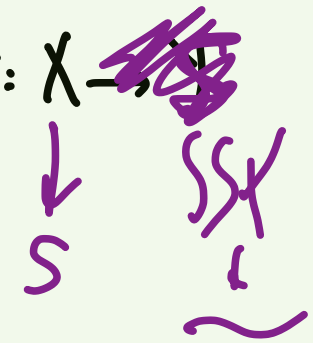
connected reduced

$s \in H^0(\bar{X}, \mathcal{O}_{\bar{X}})$

$x \in \bar{k}$

Here then is a nice class of morphisms:

$\pi: X \to S$ proper, flat, with geometrically reduced and geometrically connected fibers. $:=$ 0-connected.



This notion is: local on the base, and preserved by base change.

($:=$ "universally 0-connected")

Question (that I've not thought about, but I want to know the answer to): Is it closed under composition?