Moduli Spaces in Algebraic Geometry

Math 245 A (Winter 2022)

Feb. 14, 2022.

Theorem

$$\begin{array}{ccc} K & \xrightarrow{\phi} & \mathcal{F} \\ & \searrow \swarrow & \\ & \mathbb{P}^n_A & \\ & \downarrow \pi & \\ & \operatorname{Spec} A & \end{array}$$

$K, \mathcal{F}$ coherent

$\mathcal{F}$ flat / A.

Noetherian

Then there exists a closed subscheme $\operatorname{Spec} A/I$ such that $\alpha : T \longrightarrow \operatorname{Spec} A$ satisfies $\alpha^* \phi : \alpha^* K \to \alpha^* \mathcal{F}$

$$\begin{array}{c} \mathbb{P}^n_T \\ \downarrow \\ T \end{array}$$

is the zero map if and only if α factors through $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$.

Theorem

$$\begin{tikzcd} K \arrow[r, "\phi"] & \mathcal{F} \\ \mathbb{P}^n_A \arrow[d, "\pi"'] & \\ \operatorname{Spec} A & \end{tikzcd}$$

$K, \mathcal{F}$ coherent

$\mathcal{F}$ flat $/ A$.

Noetherian

Then there exists a closed subscheme $\operatorname{Spec} A/I$ such that $\alpha : T \to \operatorname{Spec} A$ satisfies $\alpha^*\phi : \alpha^* K \to \alpha^*\mathcal{F}$ over $\mathbb{P}^n_T$ is the zero map if and only if α factors through $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$.

Proof (Where will flatness of $\mathcal{F}$ come into it?)

For $d \geq d_0$ $\quad H^{i>0}(\mathbb{P}^n_A, \mathcal{F}(d)) = 0$

(Serre vanishing), i.e.

$$R^{i>0}\pi_* \mathcal{F}(d) = 0.$$

Also then $\pi_* \mathcal{F}(d)$ is locally free on $\operatorname{Spec} A$.

By replacing $\mathcal{F}$ by $\mathcal{F}(d_0)$, we may assume:

For $d \geq 0$, $\quad \pi_* \mathcal{F}(d)$ is locally free, and

$$R^i \pi_* \mathcal{F}(d) = 0 \text{ for } i > 0.$$

As K is coherent, $K(N)$ is globally generated for some $N>0$, so we have a surjection

$$\mathcal{O}^{\oplus M} \twoheadrightarrow K(N) \quad \text{for some } M$$

$$\Rightarrow \quad \mathcal{O}(-N)^{\oplus M} \twoheadrightarrow K \xrightarrow{\phi} \mathcal{F} \quad \text{for some } M.$$

so it suffices to deal with $\mathcal{O}^{\oplus M} \longrightarrow \mathcal{F}(N)$

hence with $\underset{K}{\overset{\parallel}{\mathcal{O}^{\oplus M}}} \xrightarrow{\phi} \mathcal{F}$ (renaming $\mathcal{F}$ again)
(renaming K)
(renaming ϕ too)

Let's find I, then show that it satisfies our universal property.

$$\phi = 0 \implies \pi_* K(d) \to \pi_* \mathcal{F}(d) \text{ is } 0 \quad \forall d \geq 0$$

$$\implies \pi_* K(d)) \to \pi_* \mathcal{F}(d) \text{ is } 0 \quad \forall d \gg 0.$$

$$\implies \phi = 0$$

Now these are maps of vector bundles. locally: maps of vector spaces. $\therefore$ get ideals I_d for when these maps are zero.

Spencer:

$$I_0 = I_1 = I_2 = \cdots$$

Hence our obvious candidate is:

$$I = I_0 + I_1 + I_2 + \cdots = I_0 = I_1 = I_2 = \cdots$$

Ben Church asks:

maybe $I = I_N + I_{N+1} + \cdots$?

Answer: Yes! If $I' = I_N + I_{N+1} + \cdots$, then on A/I', ϕ is the zero map!

We next claim that $\operatorname{Spec} A/I$ satisfies the universal property. Easy direction:

$$
\begin{tikzcd}
\mathcal{K}_T \arrow[rr, "\therefore 0"] \arrow[dr, dash] & & \mathcal{F}_T \arrow[dl, dash] & \mathcal{K}_{A/I} \arrow[rr, "0"] \arrow[dr, dash] & & \mathcal{F}_{A/I} \arrow[dl, dash] & \mathcal{K} \arrow[rr, "\phi"] \arrow[dr, dash] & & \mathcal{F} \arrow[dl, dash] \\
& \mathbb{P}^n_T \arrow[rrr] \arrow[d] \arrow[drrr, phantom, "\ulcorner", very near start] & & & \mathbb{P}^n_{A/I} \arrow[rrr] \arrow[d] & & & \mathbb{P}^n_A \arrow[d] & \\
& T \arrow[rrr] & & & \operatorname{Spec} A/I \arrow[rrr] & & & \operatorname{Spec} A &
\end{tikzcd}
$$

Harder direction:

$$\begin{tikzcd}
K_T(d) \arrow[r, "0"] \arrow[dr] & F_T(d) \arrow[d] & K(d) \arrow[r, "\phi(d)"] \arrow[dr] & F(d) \arrow[d] \\
& \mathbb{P}^n_T \arrow[rr] \arrow[d] \arrow[drr, phantom, "\ulcorner"] & & \mathbb{P}^n_A \arrow[d] \\
& T \arrow[rr] & & \operatorname{Spec} A
\end{tikzcd}$$

By our "first lemma", after a sufficiently big twist $d \gg 0$, pushforward of K, F commutes with base change, so

$$T \to \operatorname{Spec} A / I_d + I_{d+1} + \cdots = \operatorname{Spec} A/I.$$

//

Conclusion: From last day, we have proved that the Quot scheme exists in some vast generality, e.g.

$$\left\{\begin{array}{l} \mathcal{G} \ \text{ flat over } S \\ X \\ \big\downarrow \ \text{locally projective} \\ S \ \text{ locally Noetherian} \end{array}\right.$$

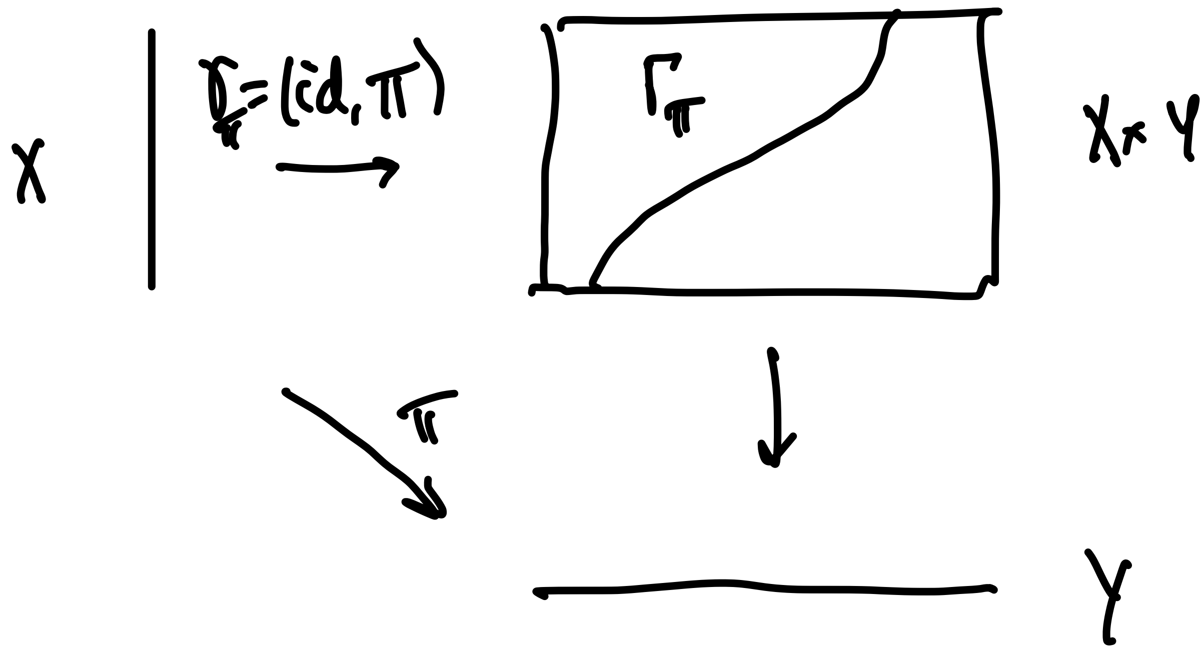
$$\begin{array}{l} \operatorname{Ker}\alpha \to \mathcal{O}(-1) \xrightarrow{\alpha} \mathcal{G} \to 0 \\ \mathbb{P}^n_S \\ \big\downarrow \\ S \end{array}$$

Moduli spaces of morphisms

The "Mor" scheme

Suppose X and Y are projective k-varieties.

Then any morphism $\pi: X \to Y$ yields a graph:



where γ_π is a closed embedding. (why?)

and if Γ_π is the image, then $pr_1 \Gamma_\pi$ is the identity.

families of morphisms should be closed subschemes of $X \times Y$ whose projections to X is an isomorphism.

Expectation:

Discuss: contravariant **FUNCTOR** : $(\underline{\text{Schemes}}_S) \longrightarrow (\underline{\text{Sets}})$

$$\mathrm{Mor}\left(\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}\right) : \left(\begin{tikzcd} B \arrow[d] \\ S \end{tikzcd}\right) \longmapsto \begin{tikzcd} X \times_S B \arrow[r] \arrow[dr] & Y \times_S B \arrow[d] \\ & B \end{tikzcd}$$

---

equivalent:

subschemes of $X \times Y$.

$$\mathrm{Mor}(X \to Y) . \ (B) \longmapsto \underbrace{\left\{ \text{closed subschemes of } X \times Y \times B \ \text{such that proj to } X \text{ is an iso} \right\}}_{\mathrm{Hilb}(X \times Y)}$$

Theorem Given

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\alpha"'] & & Y \arrow[dl, "\beta"] \\ & S & \end{tikzcd}$$

flat projective α, projective flat β

S local Noetherian

Then there is an open subset (subscheme) $U \subset S$ over which π is an isomorphism.

$U = \emptyset$

Theorem Given

$$\begin{tikzcd} X \arrow[rr, "\pi \text{ projective}"] \arrow[dr, "\alpha"'] & & Y \arrow[dl, "\beta"] \\ & S & \end{tikzcd}$$

flat projective α, β projective flat

local Noetherian

Then there is an open subset (subscheme) $U \subset S$ satisfying the following universal property:

$\gamma : T \to S$ satisfies

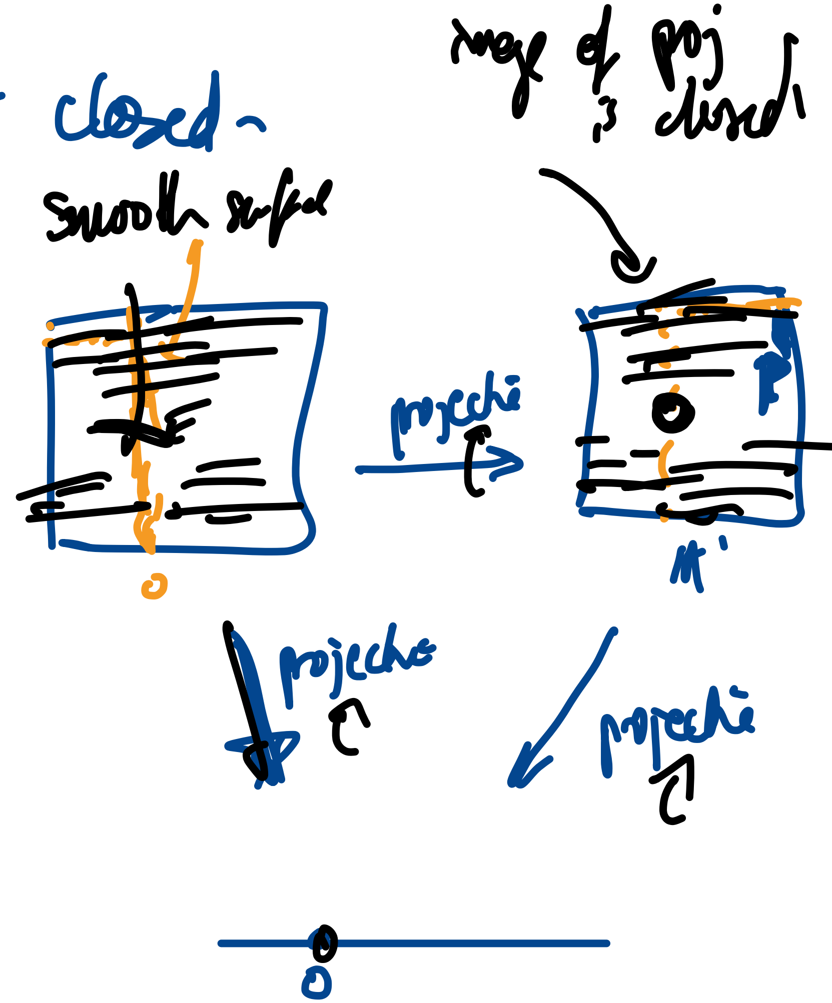
$$\begin{tikzcd} X \times_S T \arrow[rr, "\gamma^* \pi", "\sim"'] \arrow[dr] & & Y \times_S T \arrow[dl] \\ & T & \end{tikzcd}$$

if and only if γ factors through $U \underset{\text{open}}{\hookrightarrow} S$.

Proof: soon.

Example where it is open subset not closed.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\text{flat proj}"'] & & Y \arrow[dl, "\text{flat proj}'"] \\ & \mathbb{A}^1 & \end{tikzcd}$$



Definition Given

$$\begin{tikzcd} \overset{\text{flat}}{X} \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

projective

S locally Noetherian (e.g. $\operatorname{Spec}\mathbb{C}$)

Define $\operatorname{Mor}_S(X, Y)$, the scheme of morphisms from X to Y over S, as follows.

FUNCTOR (contravariant functor from S-schemes to Sets)

$$\operatorname{Mor}_S(X, Y) : (T \to S) \longmapsto \begin{tikzcd} X \times_S T \arrow[rr] \arrow[dr] & & Y \times_S T \arrow[dl] \\ & T & \end{tikzcd}$$

Theorem $\mathcal{M}or_S(X, Y)$ is representable by a scheme

$$\begin{array}{c} Mor_S(X, Y). \\ \downarrow \\ S \end{array}$$

Proof It is an open subscheme of $\mathrm{Hilb}\left(\begin{array}{c} X \times_S Y \\ \downarrow \\ S \end{array}\right)$ where the projection to X is an isomorphism. //

If $S = \mathrm{Spec}\, k$, it is a countable union of quasiprojective schemes.

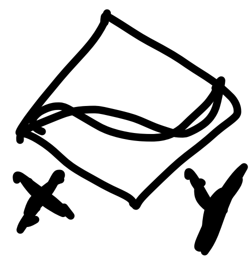
Define

$$\mathrm{Mor}\left(\begin{tikzcd}[column sep=small] X \arrow[dr, "\text{flat}"'] & & Y \arrow[dl] \\ & S & \end{tikzcd}\right)_{\text{proj}} \longrightarrow S$$

$$\boxed{\mathrm{Isom}\left(\begin{tikzcd}[column sep=small] X \arrow[dr, "\text{flat}"'] & & Y \arrow[dl, "\text{flat}"] \\ & S & \end{tikzcd}\right)_{\text{proj}}} \longrightarrow S$$

$$\mathrm{Aut}\left(\begin{tikzcd} X \arrow[d, "\text{flat}, \text{proj}"] \\ S \end{tikzcd}\right) \longrightarrow S$$

$$\mathrm{Isom}\left(\begin{tikzcd}[column sep=small] X \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}\right) \longrightarrow S$$



pr_1 iso

pr_2 iso.

$\mathrm{Isom} \xhookrightarrow{\text{open}} \mathrm{Mor} \xhookrightarrow{\text{open}} \mathrm{Hilb}.$

Automorphism scheme over S.

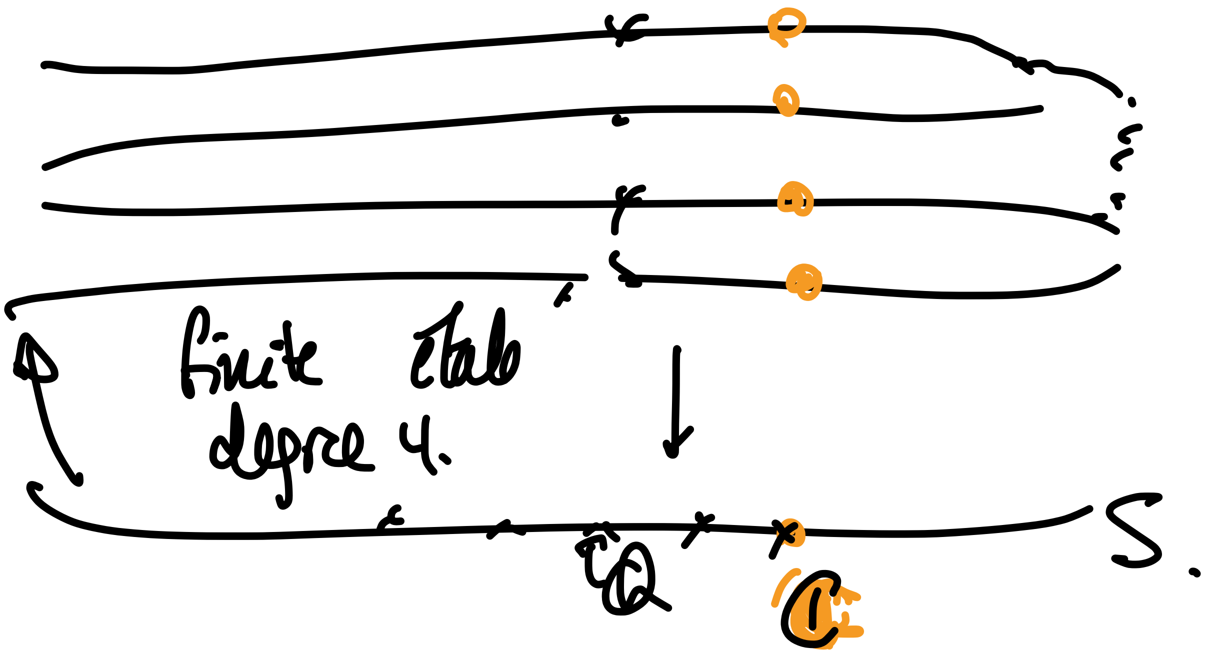
Exercise (!!!)

This is a group scheme over S.

What are the hypotheses?

What are the conclusions?

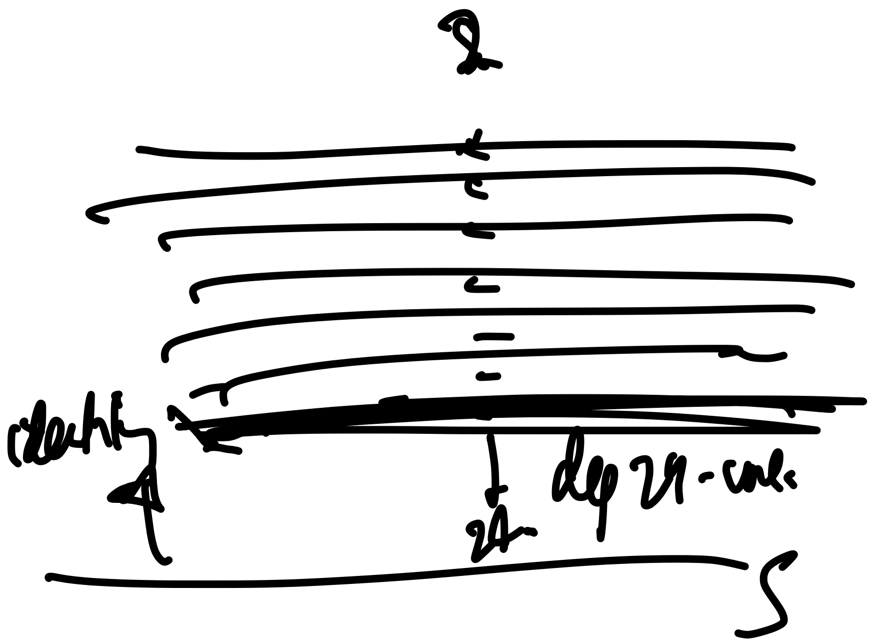
Picture of Aut. group scheme over a base



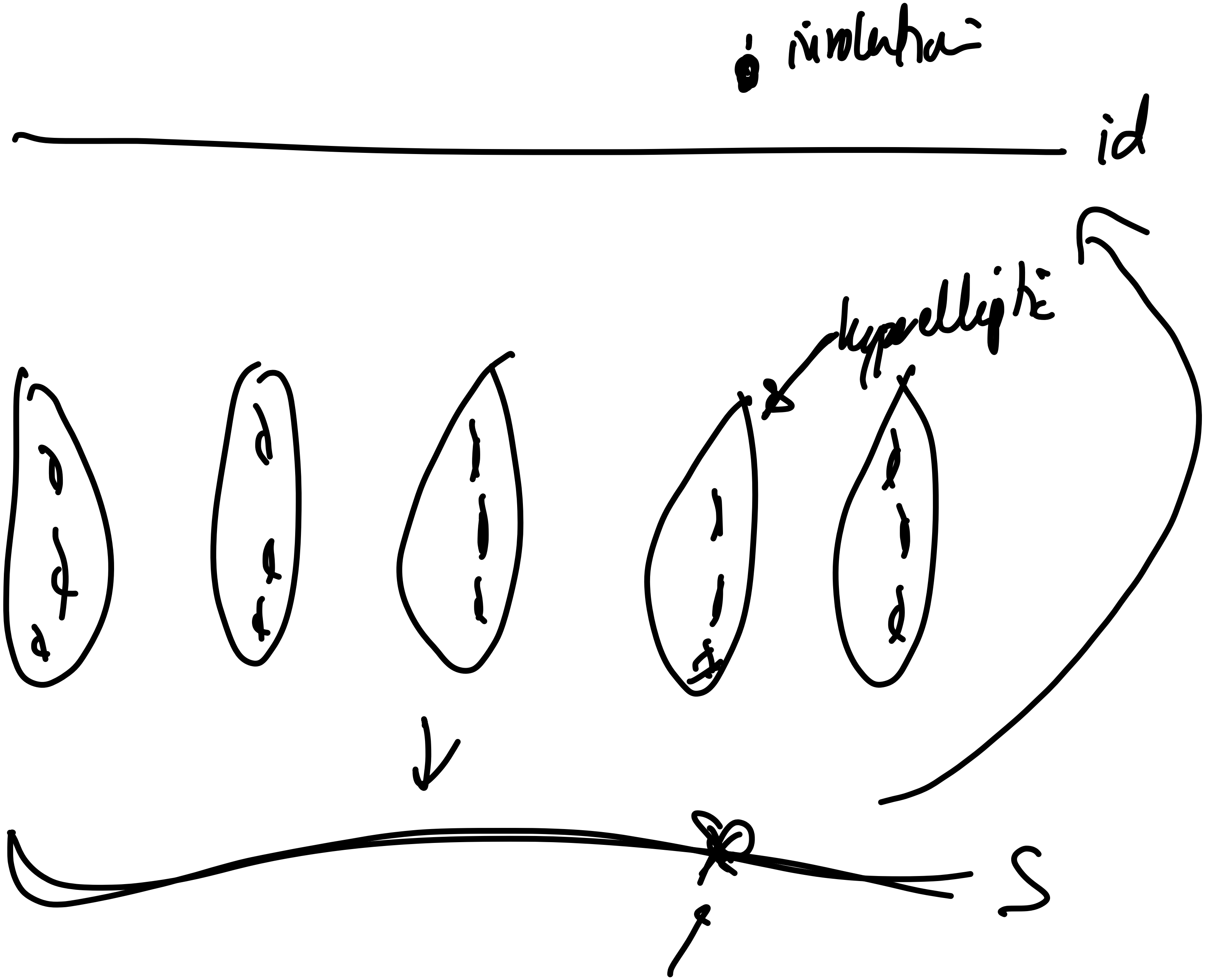
$$\begin{tikzcd} X \arrow[d] \\ S. \end{tikzcd}$$

$$\begin{tikzcd} X' \arrow[dr] & Y' \arrow[d] & X \arrow[d] & Y \arrow[dl] \\ & S' \arrow[r] & S & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Aut}\, X \arrow[d] \\ S \end{tikzcd}$$



Exercise (?)



i involution
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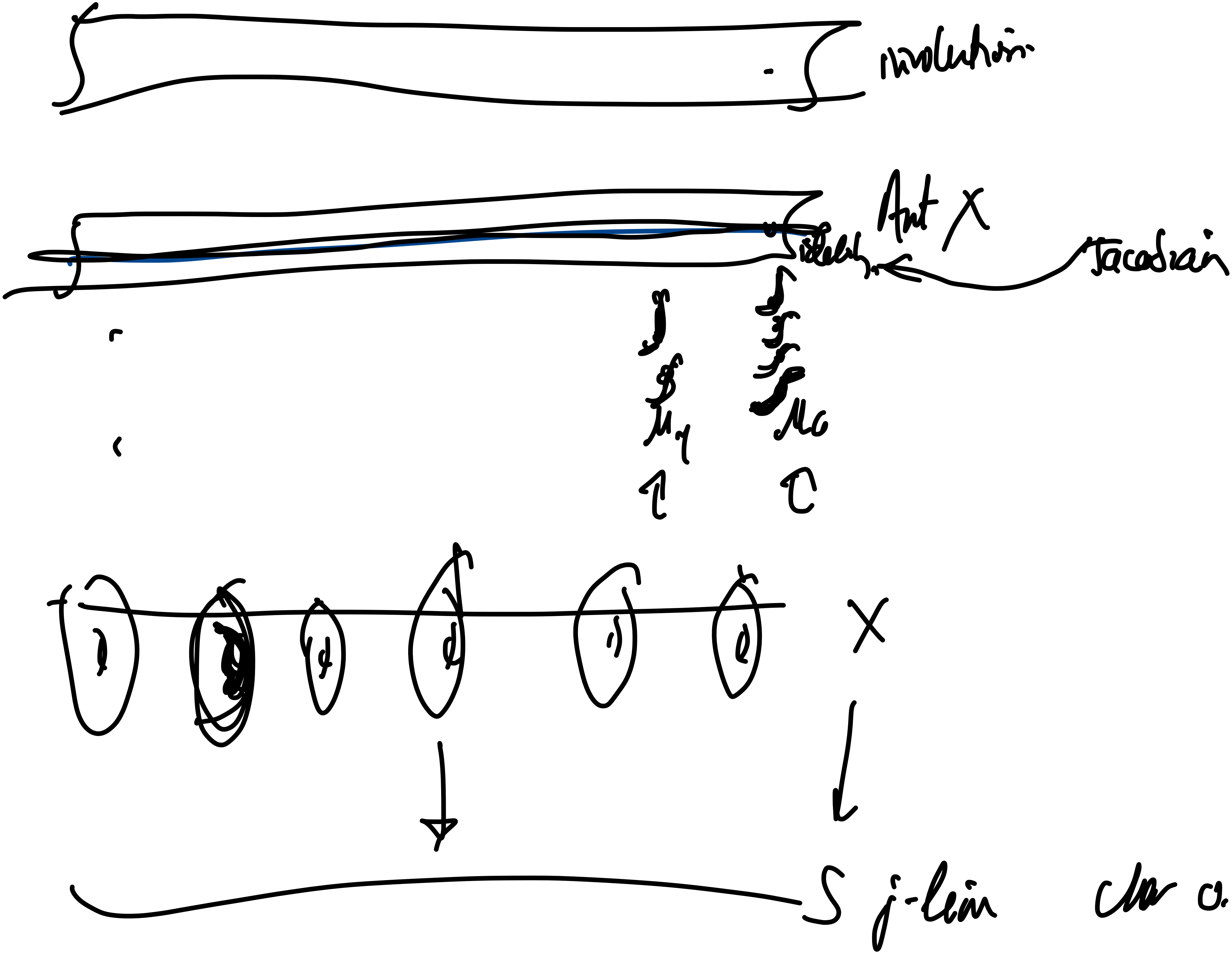

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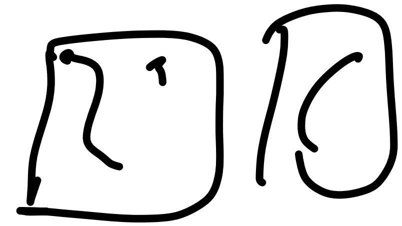
Ant C finite type scheme / $\mathbb{C}$.

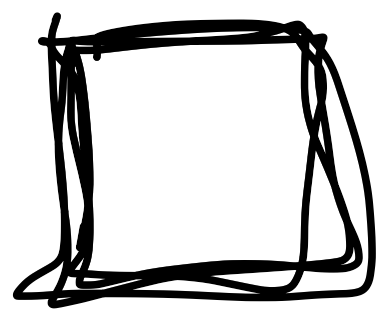


$C = \int$ genus $g > 1$.

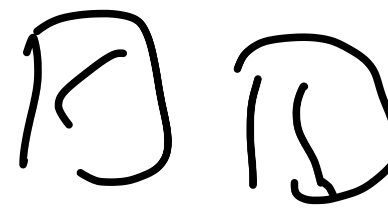
$g = 1$

$\downarrow$

$/ \mathbb{C}$.



K3 surface



Aside:

The automorphism group of any curve of genus $g > 1$ is finite (partial proof).

What do we need?

What is left to show?

Ponder: $\mathrm{Aut}_S X \curvearrowright \begin{tikzcd} X \arrow[d] \\ S \end{tikzcd}$ $\begin{tikzcd} \operatorname{Spec} \mathbb{C} \arrow[d] \\ \operatorname{Spec} \mathbb{R} \end{tikzcd}$

$\mathrm{Aut}\left(\begin{tikzcd} X \arrow[d] \\ S \end{tikzcd}\right)$.

group scheme.