

Moduli Spaces in Algebraic Geometry

Math 245 A (winter 2022)

Feb. 11, 2022.



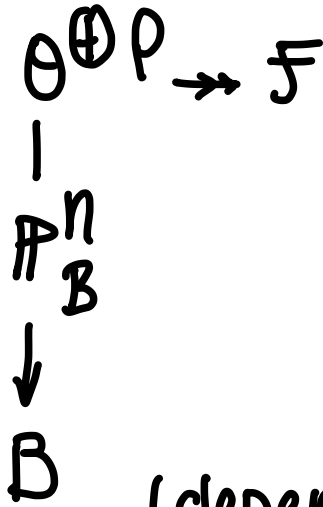
the Quot
scheme

The Quot FUNCTOR

Families of (flat) quotients of $\mathcal{O}_{\mathbb{P}^n}^{\oplus \rho}$:

Quot:
 $n, \rho, p(t)$

$B \rightsquigarrow$



flat over B

Hilbert pol. $p(t)$

(depends on $n, \rho, p(t)$)

We have proved:

Theorem (Grothendieck) This functor is representable. (This is the definition of the Quot scheme for $n, \rho, p(t)$.)

The Quot scheme is projective over $\text{Spec } \mathbb{Z}$.

quasi

It is a locally closed subscheme of a Grassmannian!

The proof involved:

Castelnuovo-Mumford regularity
(to kill cohomology)

cohomology and base change

(higher pushforwards sometimes
commute with base change)

the flattening stratification

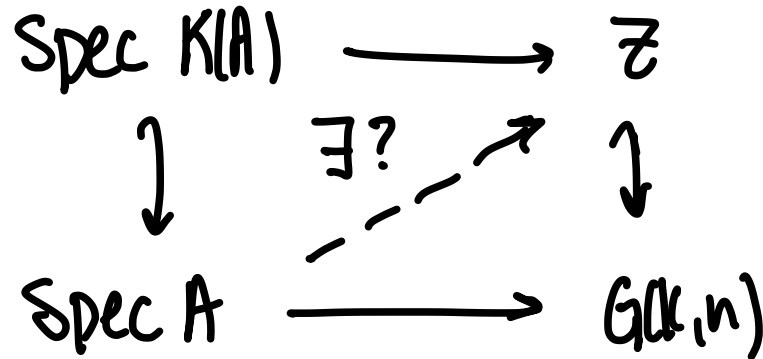
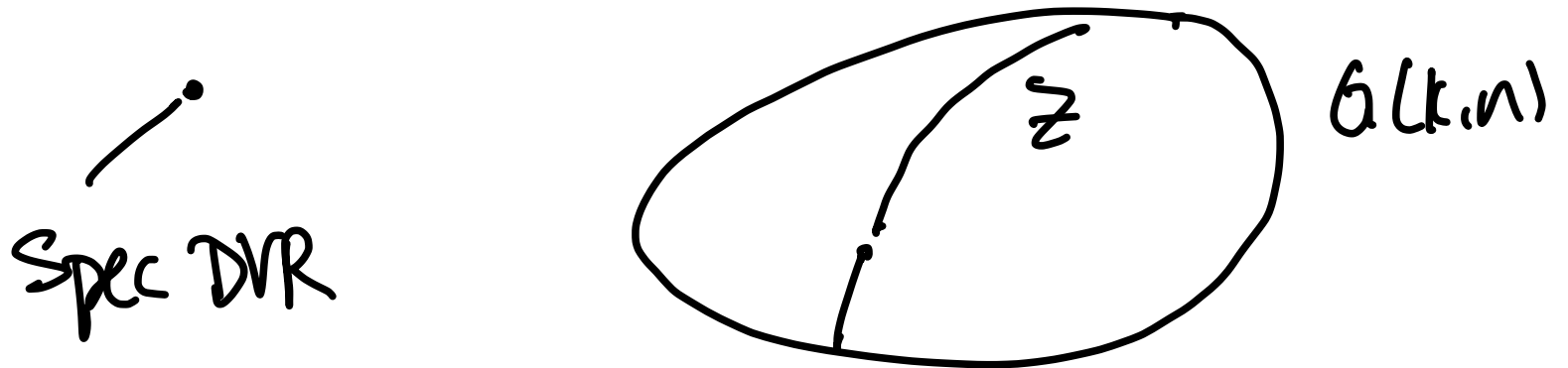
To complete our proof, we want to show that this locally closed subscheme of the Grassmannian is in fact a closed subscheme.

Someone trying to impress you: let's use the valuative criterion of properness! (Then you have to prove the valuative criterion of properness.)

Cheaper by far: To show that a locally closed subscheme is closed, you need only show that the underlying locally closed set is closed.

The ambient space is $G(k, n)$ over $\text{Spec } \mathbb{Z}$.

I present to you: the valuative criterion to check if a locally closed subset Z of $G(k, n)$ is closed.



(ok, it is the same as the valuative criterion of properness)

So we now have a question.

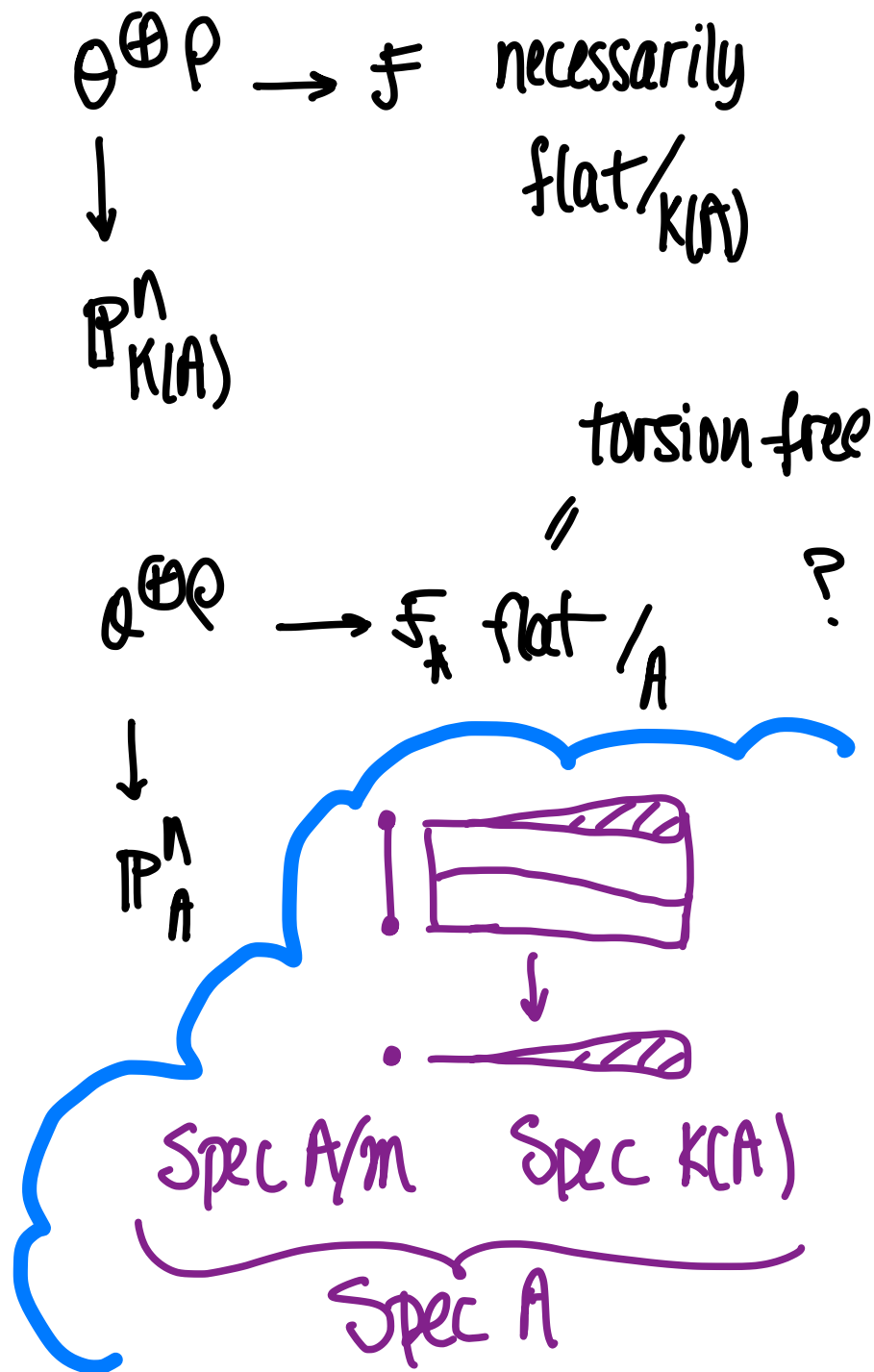
Given a DVR A , with
fraction field $K(A)$, and:

Can you extend this to:

Grothendieck says yes.

Why is he right?

(recurring theme)



$i: \text{Spec } K(A) \rightarrow \text{Spec } A$ is an affine morphism

$$\begin{array}{ccc} \text{---} & & \text{Spec } K(A) \\ \downarrow & & \\ \text{---} & \bullet & \text{Spec } A \end{array}$$

$$\mathcal{O}_{K(A)} \oplus \rho \rightarrow \mathcal{F}_{K(A)} \quad \text{coherent}$$

$$\mathcal{O}_{\mathbb{P}^n_A} \oplus \rho \rightarrow i_* \mathcal{O}_{\mathbb{P}^n_{K(A)}} \oplus \rho \rightarrow i_* \mathcal{F}_{K(A)} \quad \text{quasicoherent}$$

coherent

Let \mathcal{F}_A be the image of $\mathcal{O}_{\mathbb{P}^n_A} \oplus \rho$ in $i_* \mathcal{F}_{K(A)}$

finitely generated \Rightarrow coherent!

This extends the map over $\text{Spec } k[A]$,

to something over $\text{Spec } A$.

Furthermore: it is flat. Reason: flat = torsion-free.
for a PID

Recall (cheap — I can explain):

if A is a Principal Ideal Domain, and M is an A -module, then M is flat/ A iff M is torsion-free.

(Special case: $D = \text{Spec } k[\varepsilon]/(\varepsilon^2)$ (the "dual numbers" over k): M is flat / D iff $M/(\varepsilon M) \rightarrow \varepsilon M$ is an isomorphism — to be used in later weeks.)

We have completed the proof of :

Theorem (Grothendieck) This functor is representable. (This is the definition of the Quot scheme for $n, \rho, p(t)$.)

The Quot scheme is projective over $\text{Spec } \mathbb{Z}$.

Theorem (Grothendieck) This ^{other} functor is representable. (This is the definition of the Quot scheme for n, ρ .)

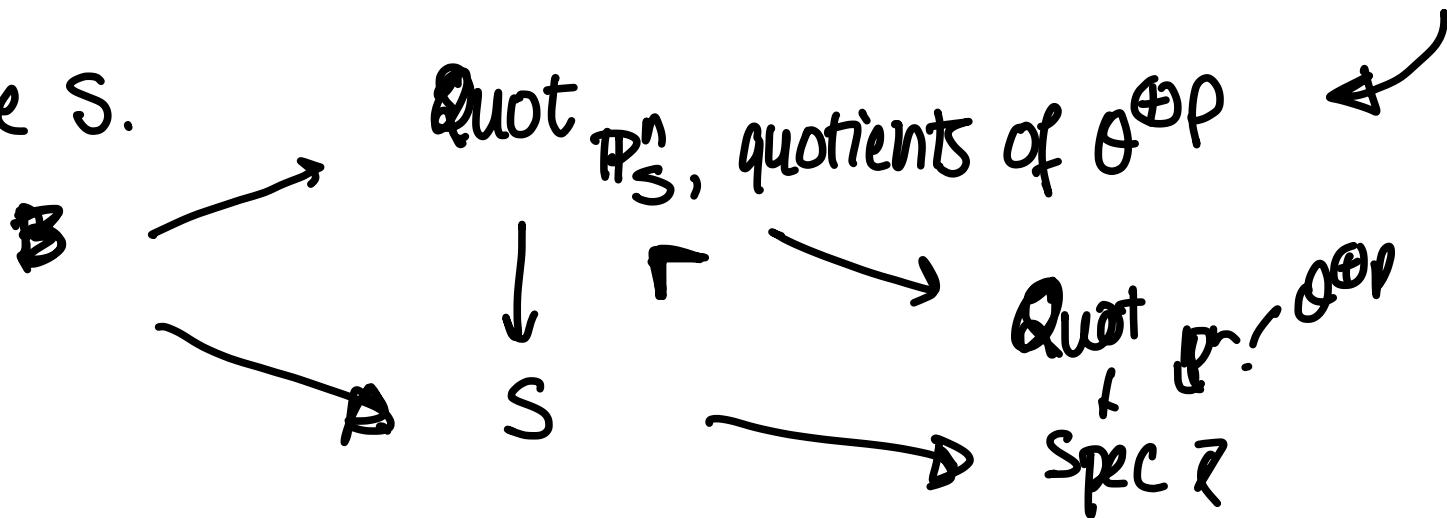


$$\text{Quot}_{\mathbb{P}^n} \mathcal{O}^{\oplus p}$$

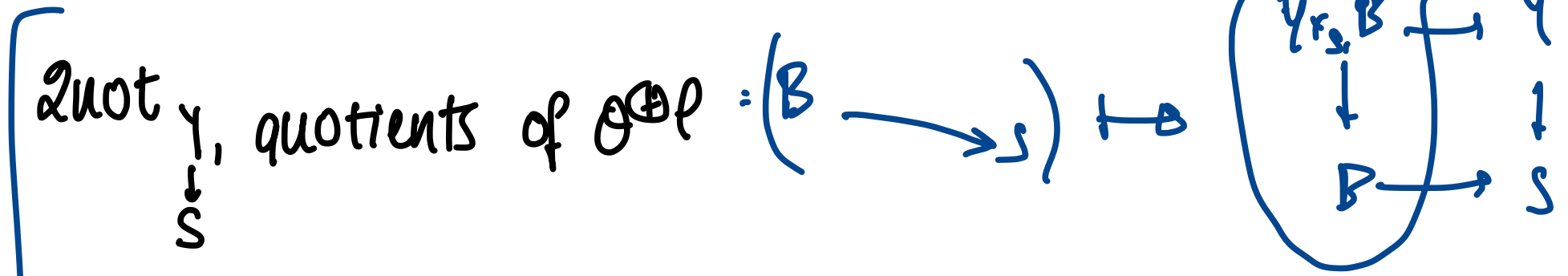
We now collect some easy consequences.

Representability of the following?

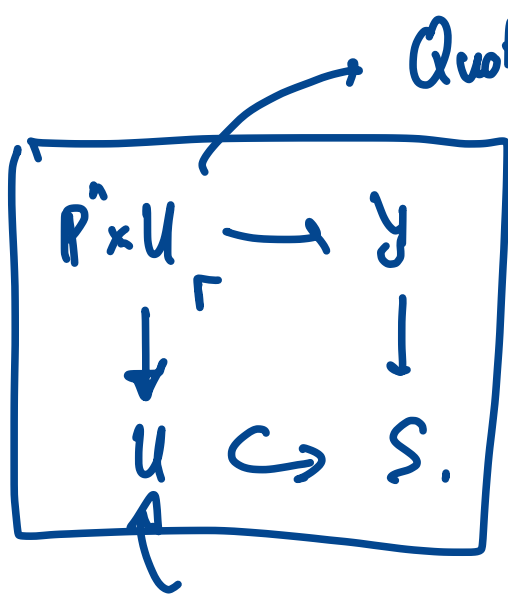
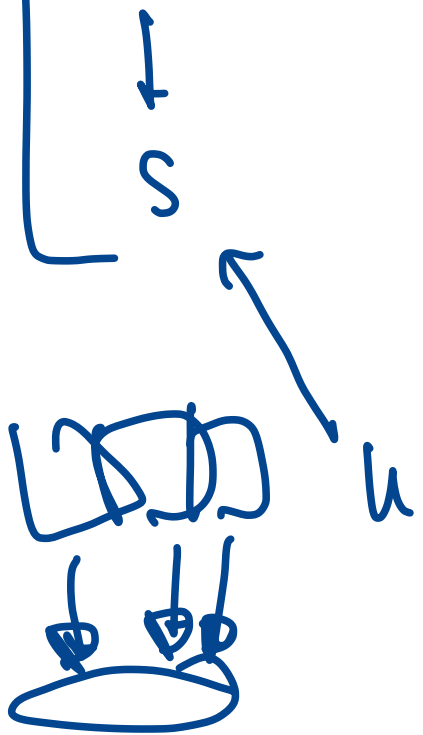
Fix scheme S .



If $Y \rightarrow S$ is a (Zariski) \mathbb{P}^n -bundle over S ,



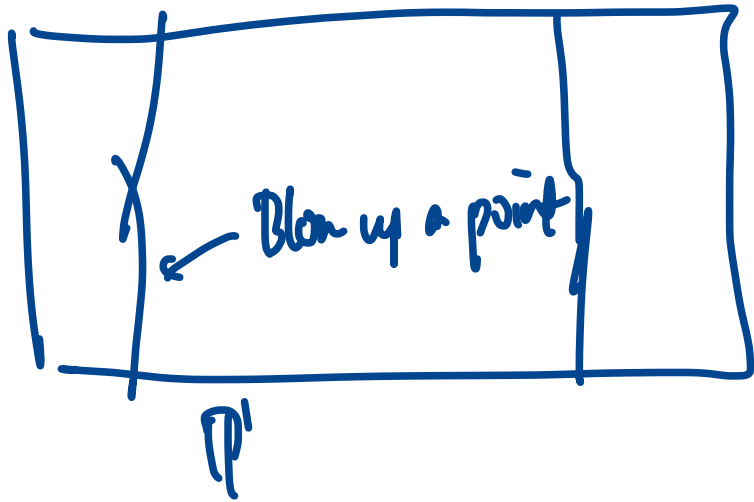
flat over B



"It is a sheaf and it is locally representable"

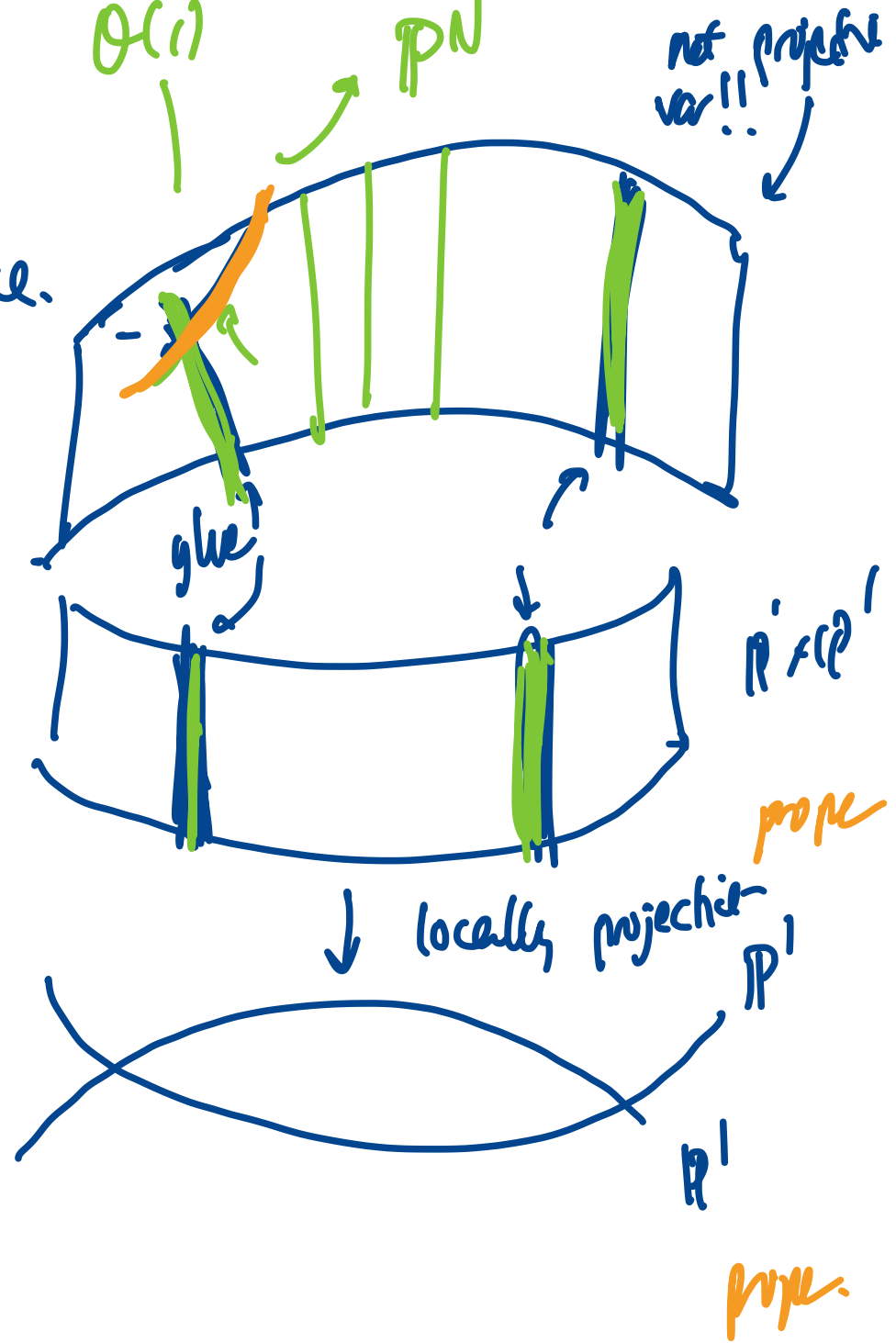
\mathbb{Z}

Important example



smooth surface.

\mathbb{R}^1

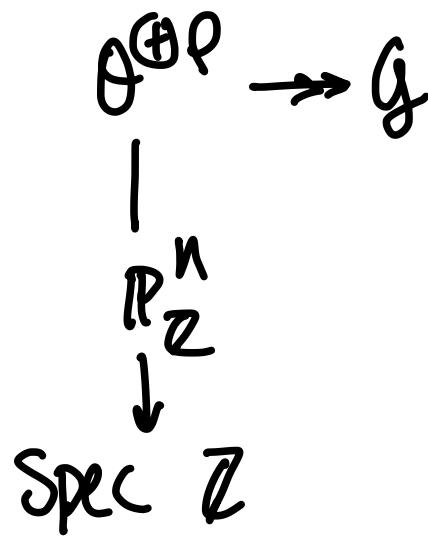


$$\text{Quot}_{\mathbb{P}^n} \mathcal{O}(3) \oplus \mathcal{O} \cong \text{Quot}_{\mathbb{P}^n} \mathcal{O} \oplus \mathcal{O}$$

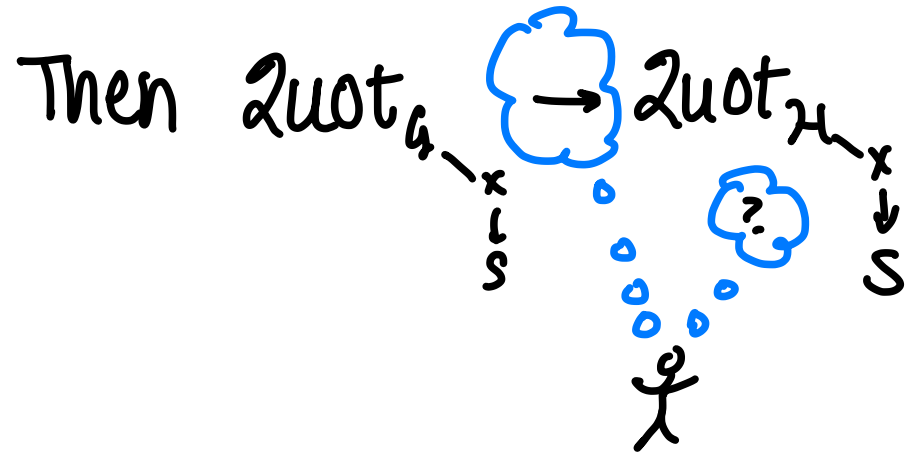
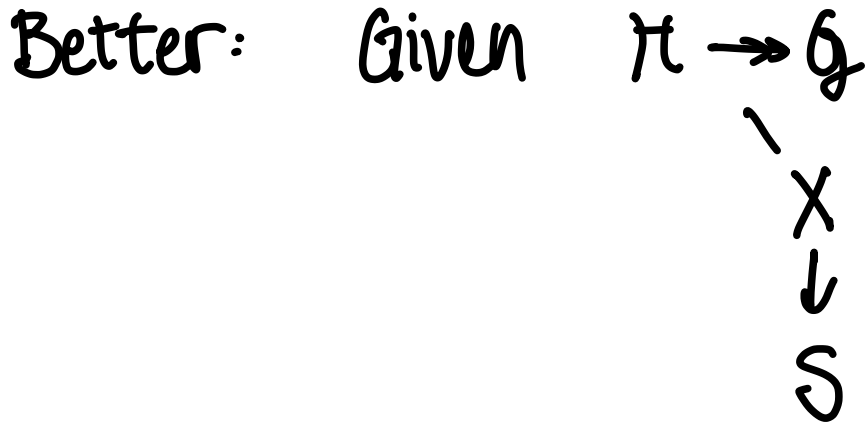
$$\begin{array}{c} \mathcal{O} \oplus \mathcal{O} \\ \downarrow \\ \mathbb{P}^n \\ \downarrow \\ B \end{array} \rightarrow \mathcal{F}(-3) \text{ flat}$$

$$B \xrightarrow{\quad \quad \quad}$$

Suppose we have:



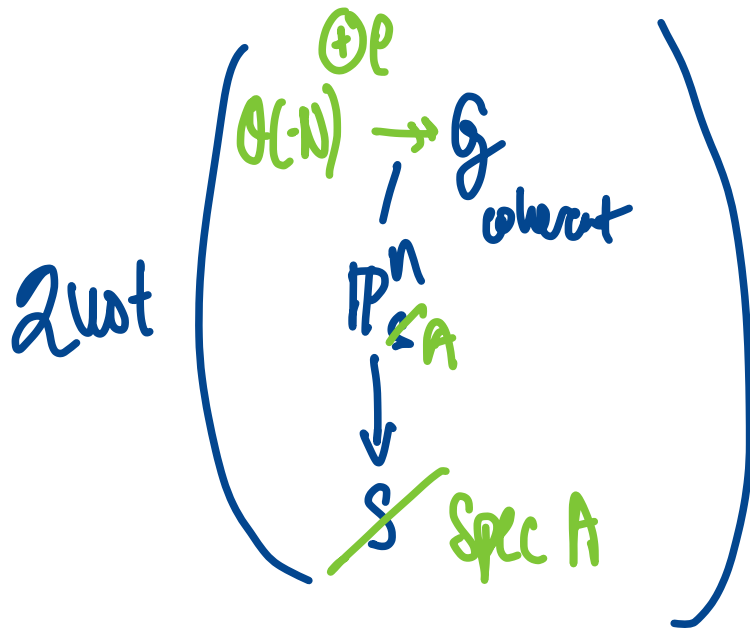
Then $\text{Quot}_{\mathbb{P}^n_{\mathbb{Z}}/\mathcal{G}}$
 \downarrow
 $\text{Spec } \mathbb{Z}$
 is representable.



hypotheses?

is a closed subfunctor
 why?

If we can do this, we'll know that



is representable!

(Reduction steps
in green)

Let $K = \ker(\alpha - \beta)$.

$$K \rightarrow F$$

\downarrow

\downarrow

S

The locus where this is the zero map is a closed subscheme of S .

Let's make this precise.

We will use: F flat.

S noetherian. K, F coherent.

X projective.

Closed subscheme defined by universal property.

Theorem

$$\begin{array}{c}
 K \xrightarrow{\phi} \mathcal{F} \\
 \downarrow \text{IP}_A^n \\
 \downarrow \pi \\
 \text{Spec } A
 \end{array}$$

K, \mathcal{F} coherent
 \mathcal{F} flat / A .

Then there exists a closed subscheme $\text{Spec } A/I$ such that $\alpha: T \rightarrow \text{Spec } A$ satisfies $\alpha^* \phi: \alpha^* K \rightarrow \alpha^* \mathcal{F}$

$$\begin{array}{c}
 \downarrow \\
 \text{IP}_T^n
 \end{array}$$

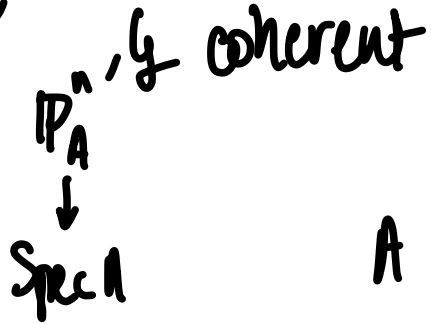
is the zero map if and only if α factors through

$$\text{Spec } A/I \hookrightarrow \text{Spec } A.$$

Proof:

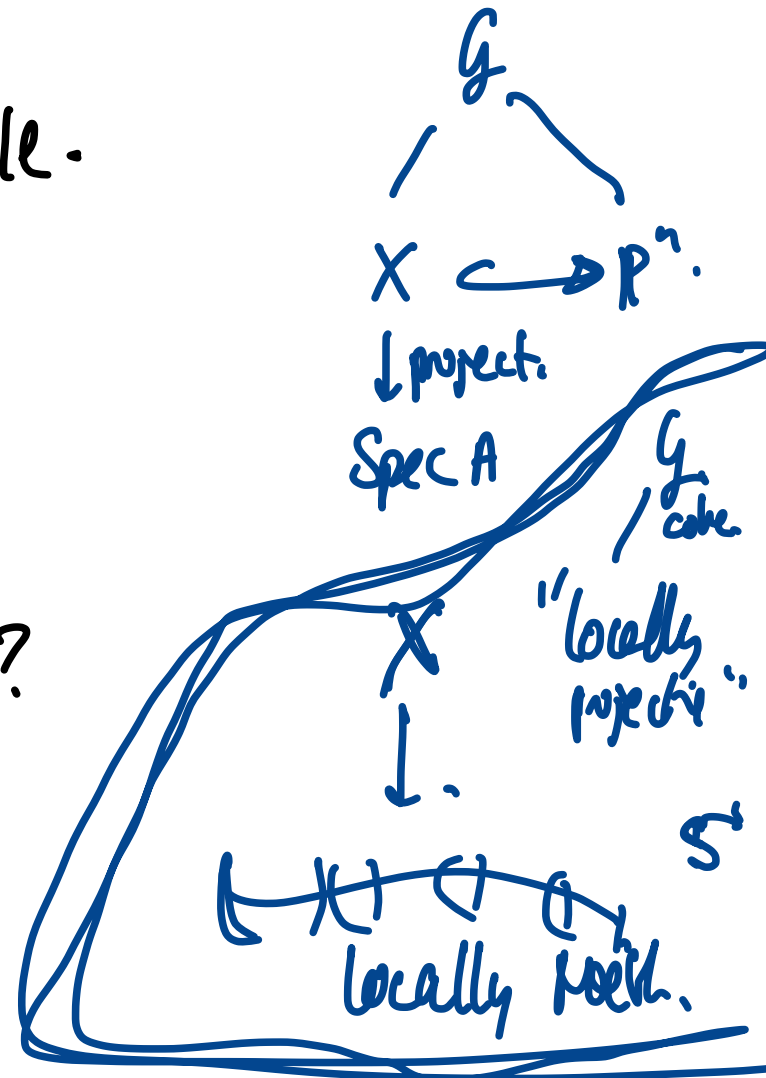
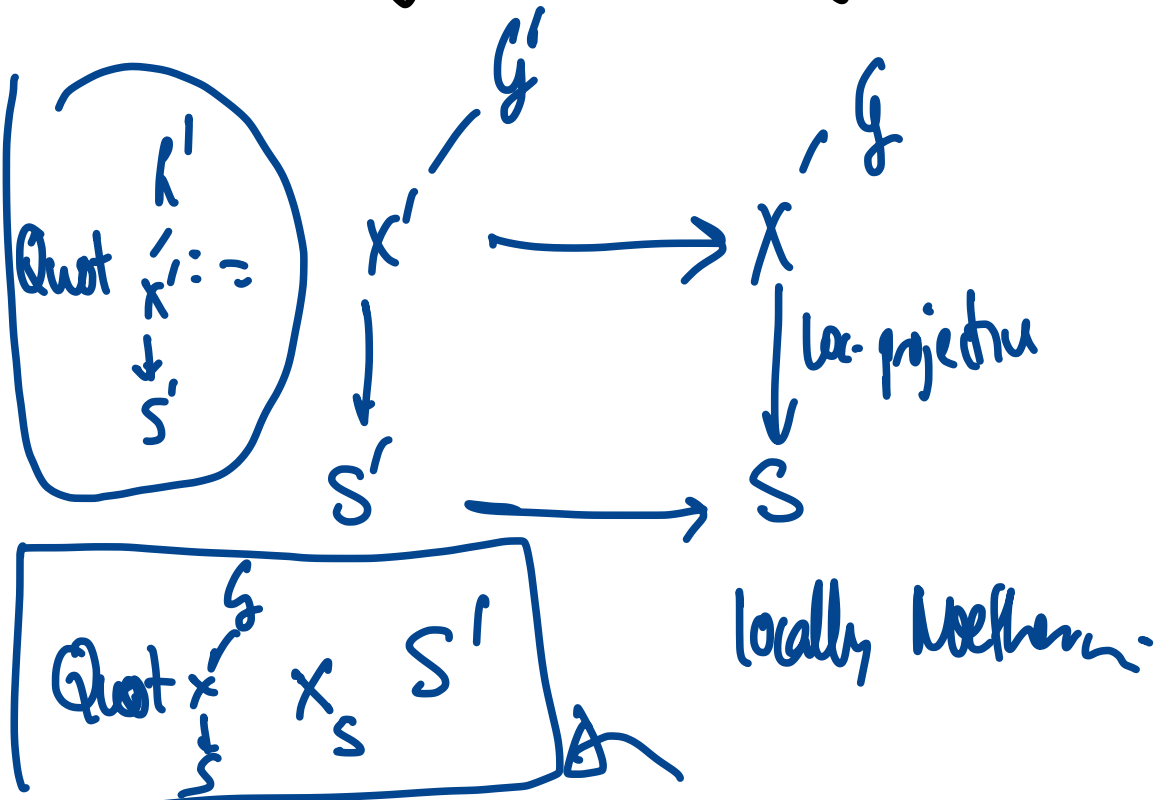
What will you do?

Quot $\mathbb{P}^n, \mathcal{G}$ coherent is representable.



A Noetherian

How general can you make this?



(non-Noetherian?)

Even if S is not nec. Noetherian.

defn: $\text{Quot } X / \mathcal{G}$ *hypothesis. finitely presented
 \downarrow bc projective finitely presented
 S

parametrize flat quotients of \mathcal{G} . *hypothesis. finitely presented.

\exists $X \hookrightarrow \mathbb{P}_S^n$ "degree d hypersurface."
 flat \searrow S
 non-Noetherian. affine
Not finitely presented
 \therefore does not arise from \mathcal{U} -f.p.
 $S \rightarrow \mathbb{P}(\binom{n+d}{n})-1$

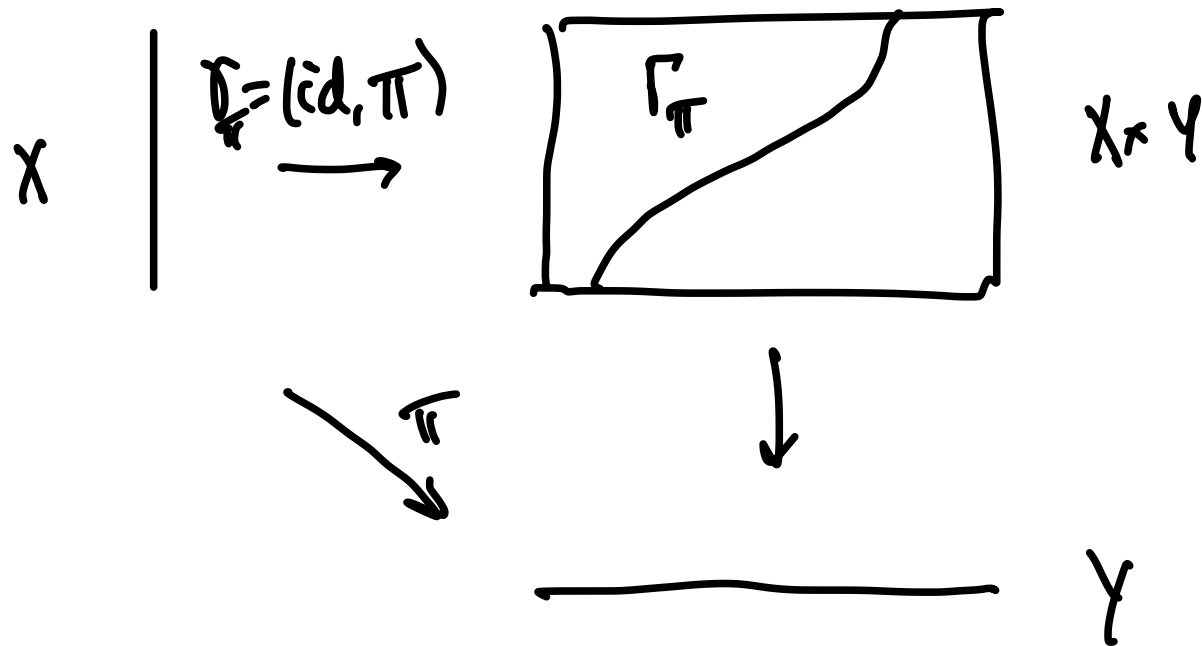
Then: Hilbert scheme of X in \mathbb{P}^1 .

Hilbert scheme of varying X !

The "Mor" scheme

Suppose X and Y are projective k -varieties.

Then any morphism $\pi: X \rightarrow Y$ yields a graph:



where δ_π is a closed embedding. (why?)

and if Γ_π is the image, then $\text{pr}_1 \Gamma_\pi$ is the identity.

Expectation:

families of morphisms should be closed
subschemes of $X \times Y$ whose projections
to X is the identity.

Discuss:

Lemma

Given a family of morphisms, the locus where it is an isomorphism is an open subset.

What should the precise statement be?

(Proof next time.)

Define If $\begin{array}{ccc} X & \rightarrow & Z \\ Y & \rightarrow & Z \end{array}$ are a projective morphism,

define $\text{Isom}_{X, Y}$.



Define $\text{Aut}_{X \rightarrow Z}$, a group scheme over Z , locally of finite type.