Moduli Spaces in Algebraic Geometry

Math 245 A (Winter 2022)

Feb. 9, 2022.

the Quot scheme

The Quot FUNCTOR

Families of (flat) quotients of $\mathcal{O}_{\mathbb{P}^n}^{\oplus p}$: here is the contravariant FUNCTOR from (Schemes) to (Sets) that we'll see is representable:

$$\mathrm{Quot}_{n,p}: \quad B \rightsquigarrow \begin{array}{c} \mathcal{O}^{\oplus p} \twoheadrightarrow \mathcal{F} \\ | \\ \mathbb{P}^n_B \\ \downarrow \\ B \end{array}$$

flat over B

(depends on n, p)

Observations:

(i) If you wish:

flat over B

$$B \rightsquigarrow \begin{array}{c} 0 \to \mathcal{E} \to \mathcal{O}^{\oplus p} \to \mathcal{F} \to 0 \\ | \\ \mathbb{P}^n_B \\ \downarrow \\ B \end{array}$$

(ii) Special cases:

$p = 1$: Hilbert scheme

$n = 0$: Grassmannian

flat over B

$$B \rightsquigarrow \begin{tikzcd} 0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O}^{\oplus p} \arrow[r] \arrow[d, dash] & \mathcal{F} \arrow[r] & 0 \\ & & \mathbb{P}^n_B \arrow[d] & & \\ & & B & & \end{tikzcd}$$

(iii) breaks up into subFUNCTORs that are simultaneously open and closed, indexed by the Hilbert polynomial $\chi(\mathbb{P}^n, \mathcal{F}(t)) =: P_{\mathcal{F}}(t)$ of $\mathcal{F}$ or, equivalently, the Hilbert polynomial $\chi(\mathbb{P}^n, \mathcal{E}(t)) =: P_{\mathcal{E}}(t)$ of $\mathcal{E}$:

$$\mathrm{Quot}_{p,n} = \coprod \mathrm{Quot}_{p,n,P_{\mathcal{F}}(t)}$$

$$\begin{array}{c} \text{flat over } B \\ 0 \to \mathcal{E} \to \mathcal{O}^{\oplus p} \to \mathcal{F} \to 0 \\ | \\ \mathbb{P}^n_B \\ \downarrow \\ B \end{array}$$

$B \rightsquigarrow$

Fix $p \in B$

$$P_{\mathcal{E}}(t) = \chi(\mathcal{E}(t))$$

Reality check: How are $P_{\mathcal{E}}(t)$ and $P_{\mathcal{F}}(t)$ related?

$$P_{\mathcal{E}}(t) + P_{\mathcal{F}}(t) = \chi(\mathcal{O}^{\oplus p}(t)) = p\,\chi(\mathcal{O}_{\mathbb{P}}(t))$$

$$= p\binom{n+t}{n} = \frac{p}{n!}(t+n)(t+n-1)\cdots(t+1)$$

Reality check: what is the Hilbert polynomial in the case $n=0$,

$$0 \to \mathcal{E} \to \mathcal{O}^{\oplus p} \to \mathcal{F} \to 0$$

$\mathcal{F}$: rank k

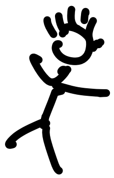
i.e., the Grassmannian?

$\mathcal{E}$: $p-k$. $\quad \mathcal{O}^{\oplus p}$: p $\quad \mathcal{F}$: k

constant.

For fixed $n \in \mathbb{Z}^{\geq 0}$, $\rho \in \mathbb{Z}^{>0}$, $\overset{= P_F(t)}{p(t)} \in \mathbb{Q}[t]$

Think: $\mathbb{P}^n$ $\quad$ $\mathcal{O}^{\oplus \rho}$ $\quad$ Hilbert polynomial



Define the corresponding $\mathcal{Q}uot_{n,\rho,p(t)}$ FUNCTOR.

<u>Theorem (Grothendieck)</u> This functor is representable. (This is the definition of the Quot scheme for n, ρ, $p(t)$.)

The Quot scheme is projective over Spec $\mathbb{Z}$.

My ~~plan~~ hope: you prove it yourself.

Let's think about how we would prove this.

Over a field k, suppose we have an object (of $\mathcal{Q}uot(\operatorname{Spec} k)$):

$$\begin{array}{c} 0 \to \mathcal{E} \to \mathcal{O}^{\oplus p} \to \mathcal{F} \to 0 \\ | \\ \mathbb{P}^n_k \end{array}$$

where $\mathcal{E}$ and $\mathcal{F}$ have known Hilbert polynomial. Thanks to Mumford*, we know there is some M (depending only on n, p, $p(t)$) such that ($\mathcal{E}$ and) $\mathcal{F}$ is M-regular. How will we use this?

* our discussion of Castelnuovo-Mumford regularity

Hence for any family in $\mathrm{Quot}^{n,p,P(t)}(B)$,

$$0 \to \mathcal{E} \to \mathcal{O}^{\oplus p} \to \mathcal{F} \to 0$$

$$\mathbb{P}^n_B \to B$$

for $m \geq M_1$ we feel like twisting by m:

$$0 \to \mathcal{E}(m) \to \mathcal{O}(m)^{\oplus p} \to \mathcal{F}(m) \to 0$$

$$\mathbb{P}^n_B \to B$$

All higher coho vanish!

$$0 \to \bigcirc \to \bigcirc \to \bigcirc \to 0$$

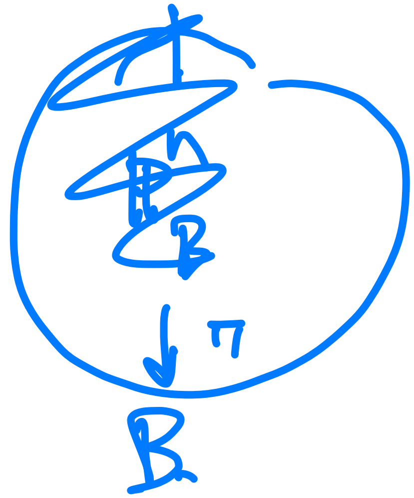
$$\mathbb{P}^n_k \to \operatorname{Spec} k$$

Then what?

locally free $\quad P_{\mathcal{E}}(m) \quad$ TRIVIAL Part $P\binom{n+m}{n}$ $\quad P_{\mathcal{F}}(m)$

$$0 \to \pi_{*}\mathcal{E}(m) \to \pi_{*}\underbrace{\mathcal{O}^{\oplus P}(m)} \to \pi_{*}\mathcal{F}(m) \to 0$$

All higher coho of fiber is zero.



$$R^1\pi_{*}\mathcal{E}(m) \to \cdots$$

$0 \quad 0$

cohomology + base change

zeroth cohomology? ON FIBERS

for each pt of B

$h^0(\mathcal{F}(m)) = P_{\mathcal{F}}(m) = \chi(\mathcal{F}(m))$,

$h^0(\mathcal{E}(m)) = P_{\mathcal{E}}(m)$

$h^0(\mathcal{O}^{\oplus P}(m)) = P\binom{n+m}{n}$

For each point $q \in B$,

$$H^{i>0}(\mathbb{P}^n, \mathcal{E}(m)) = H^{i>0}(\mathbb{P}^n, \mathcal{O}(m)^{\oplus \rho})$$
$$= H^{i>0}(\mathbb{P}^n, \mathcal{F}(m)) = 0$$

$\pi_* \mathcal{F}(m)$ is a locally free sheaf of rank $p(m)$.

We have an exact sequence of locally free sheaves on B.

$$\begin{tikzcd}
0 \arrow[r] & \pi_* \mathcal{E}(m) \arrow[r] & (\pi_* \mathcal{O}(m))^{\oplus \rho} \arrow[r] \arrow[d, dash] & \pi_* \mathcal{F}(m) \arrow[r] & 0 \\
 & & B & &
\end{tikzcd}$$

Why?

For each point $q \in B$,

$$H^{i>0}(\mathbb{P}^n, \mathcal{E}(m)) = H^{i>0}(\mathbb{P}^n, \mathcal{O}(m)^{\oplus \rho})$$
$$= H^{i>0}(\mathbb{P}^n, \mathcal{F}(m)) = 0$$

$\pi_* \mathcal{F}(m)$ is a locally free sheaf of rank $p(m)$.

We have an exact sequence of locally free sheaves on B.

$$\boxed{0 \to \pi_* \mathcal{E}(m) \to (\pi_* \mathcal{O}_{\mathbb{P}^n_B}(m))^{\oplus \rho} \to \pi_* \mathcal{F}(m) \to 0}$$

This gives a map $B \to G\underbrace{\left(p_{\mathcal{E}}(m), \rho\binom{m+n}{m}, p_{\mathcal{F}}(m)\right)}_{\text{(explain)}}$

Thus we have $\mathrm{Quot}_{\rho, n, p(t)}$

Given

$$0 \to \pi_* \mathcal{E}(m) \to (\pi_* \mathcal{O}(m))^{\oplus e} \to \pi_* \mathcal{F}(m) \to 0 \quad (*)$$

(i.e. $0 \to V \to \pi_* \mathcal{O}(m)^{\oplus e} \to W \to 0$, along with the knowledge that it is isomorphic to some $(*)$),

I claim we can recover $\mathcal{F}$ from this information. $\mathcal{E} \to \mathcal{O}^{\oplus e}$

$\therefore$ Quot is a subFUNCTOR of the Grassmannian

CAN Recover!

$$0 \to V \to \pi_* \mathcal{O}(m)^{\oplus e} \to W \to 0$$

$P_{\mathcal{E}}(m)$ | B | $\mathbb{P}^n$ | rank. $P_{\mathcal{F}}(m)$

$$0 \to \boxed{\mathcal{E} \to \mathcal{O}^{\oplus e}} \to \mathcal{F} \to 0$$

(if one exists)

$\mathcal{E} \to \mathcal{O}^{\oplus e}$ (B affine)

graded modules $\mathcal{O} = B[x_0, \dots, x_n]$

$M \hookrightarrow B[x_0, \dots, x_n]$.

$$0 \to \mathcal{E} \to \mathcal{O}^{\oplus e} \to \mathcal{F} \to 0$$

Given

$$0 \to \pi_* \mathcal{E}(m) \to (\pi_* \mathcal{O}(m))^{\oplus e} \to \pi_* \mathcal{F}(m) \to 0 \quad (*)$$

(i.e. $0 \to \mathcal{V} \to \pi_* \mathcal{O}(m)^{\oplus e} \to \mathcal{W} \to 0$, along with the knowledge that it is isomorphic to some $(*)$),

I claim we can recover $\mathcal{F}$ ($\mathcal{E} \to \mathcal{O}^{\oplus e}$) from this information.

$\therefore$ Quot is a subFUNCTOR of the Grassmannian

CAN Recover?

$$0 \to \mathcal{V} \to \pi_* \mathcal{O}(m)^{\oplus e} \to \mathcal{W} \to 0$$

rank. $P_{\mathcal{E}}(m)$, B, $P_{\mathcal{F}}(m)$

$$0 \to \boxed{\mathcal{E} \to \mathcal{O}^{\oplus e}} \to \mathcal{F} \to 0$$

(if one exists)

$\mathcal{E} \to \mathcal{O}^{\oplus e}$ (B affine)

graded modules $\mathcal{O} = B[x_0, \dots, x_n]$

$N \hookrightarrow B[x_0, \dots, x_n]$.

Fix m.

$$N_m \hookrightarrow \left(B[x_0, \dots, x_n]^{\oplus e}\right)_m$$

$$H^0(\mathcal{V}) \hookrightarrow H^0(\pi_* \mathcal{O}(m)^{\oplus e}).$$

mult. by linear form in $x_0, \dots, x_n$ to get N_{m+1}.

$\mathcal{E}(m+1)$.

?

$N_m \oplus N_{m+1} \oplus N_{m+2} \oplus \cdots$

Recover $\mathcal{E}$

Given

$$0 \to \pi_* \mathcal{E}(m) \to \left(\pi_* \mathcal{O}(m)\right)^{\oplus \ell} \to \pi_* \mathcal{F}(m) \to 0 \quad (*)$$

(i.e. $0 \to V \to \pi_* \mathcal{O}(m)^{\oplus \ell} \to W \to 0$, along with the knowledge that it is isomorphi to some $(*)$),

I claim we can recover $\mathcal{F}$ from this information.

$\therefore$ Quot is a subFUNCTOR of the Grassmannian

I also claim it is a locally closed subFUNCTOR of the Grassmannian.

Given $0 \to V \to \pi_* \mathcal{O}(m)^{\oplus \ell} \to W \to 0$.

How can you tell if it in π_* of $0 \to \mathcal{E}(m) \to \mathcal{O}^{\oplus \ell}(m) \to \mathcal{F}(m) \to 0$?

$\bigoplus_{m' \geq 0} \mathcal{W}_{m'}$ graded module / B

"$\pi_* \mathcal{F}(m+m')$".

$0 \to \mathcal{Y} \to \bigoplus \pi_* \mathcal{O}(m) \to \mathcal{W} \to 0.$

coherent sheaf over $\mathbb{P}^n_B$. $\pi_* \mathcal{O}(1)$ $\mathbb{P}^n_G$.

Quotient of $\mathcal{O}^{\oplus e}$.

$G(\vdots \; , \; \vdots \; , \; \vdots \;)$

locus: where it is flat over B with Hilbert polynomial $P_{\mathcal{F}}(t)$.

Goal: $0 \to \mathcal{E} \to \tilde{S}_0^{\,e} = \mathcal{O}^{\oplus e} \to \mathcal{F} \to 0$

flattening stratification

quotient

$0 \to$ "smallest graded S_0-submodule containing V." $\to (S_0)^e \longrightarrow$ quotient $\to 0$

$\mathbb{P}^n_G$

$0 \to V \to \bigoplus^e \pi_* \mathcal{O}_{\mathbb{P}^n_G}(m) \longrightarrow W \longrightarrow 0$

$S_0 = \bigoplus_{j \geq 0} \pi_* \mathcal{O}_{\mathbb{P}^n_G}(j)$

$G_r\left(?, \; e\binom{m+n}{n}, \; ? \right)$

$P_{\mathcal{E}}(m)$ $\qquad$ $P(m)$ $\mathcal{F}$

We have proved:

Theorem (Grothendieck) This functor is representable. (This is the definition of the Quot scheme for n, ρ, $p(t)$.)

The Quot scheme is projective over $\operatorname{Spec} \mathbb{Z}$.

quasi

It is a locally closed subscheme of a Grassmannian!

To complete our proof, we want to show that this locally closed subscheme of the Grassmannian is in fact a closed subscheme.

Someone trying to impress you: let's use the valuative criterion of properness! (Then you have to prove the valuative criterion of properness.)

Cheaper by far: To show that a locally closed subscheme is closed, you need only show that the underlying locally closed set is closed.

The ambient space is $G(k,n)$ over $\operatorname{Spec} \mathbb{Z}$.