Moduli Spaces in Algebraic Geometry

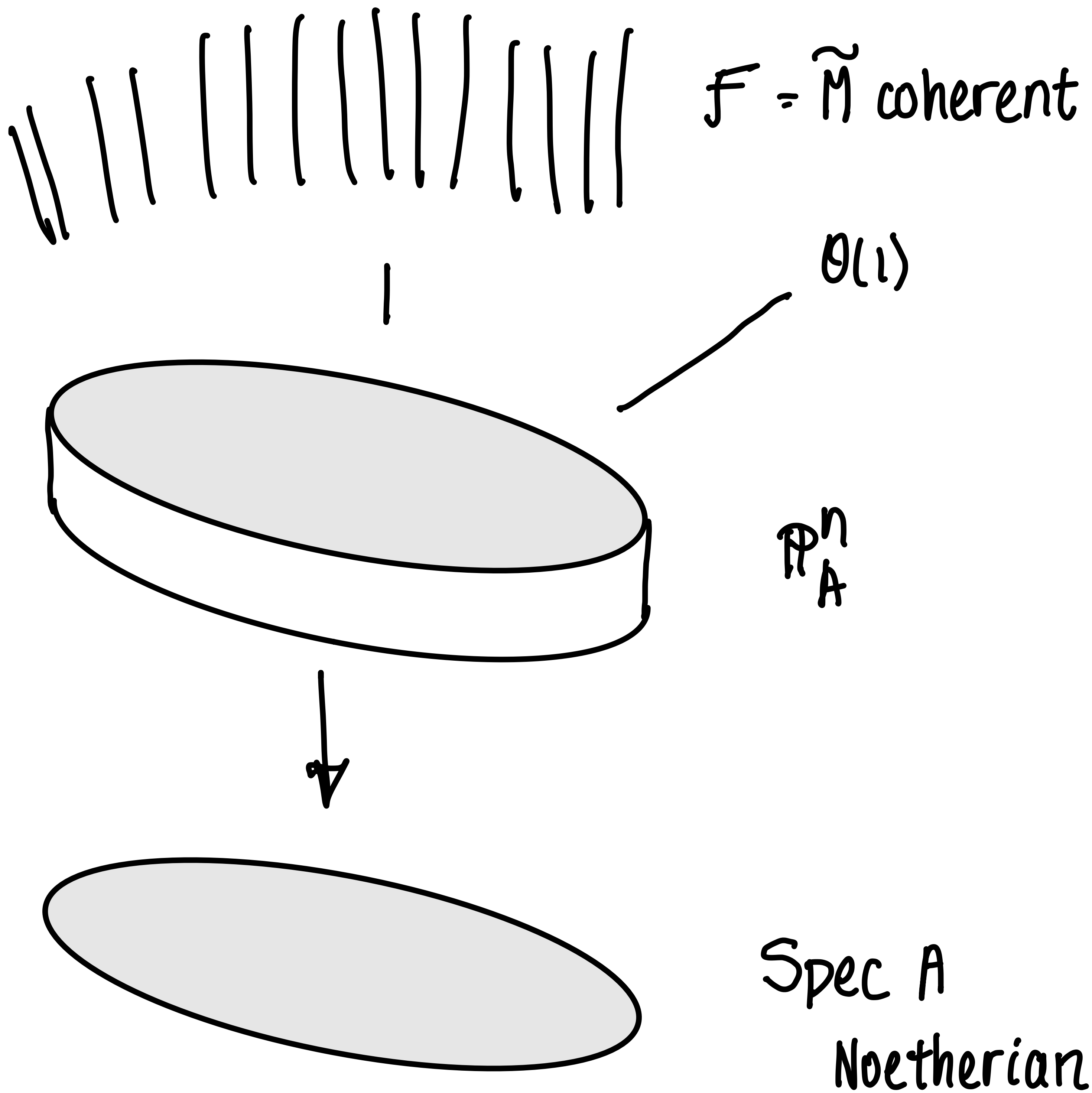
Math 245 A (winter 2022)

Feb. 7, 2022.

the flattening stratification

Ben Church saves the day

The Situation



Our goal:

Stratify into "best" locally closed subschemes over which $\mathcal{F}$ is flat

PROOF OF THE FLATTENNING STRATIFICATION:

First few steps get us to:

higher pushforward

If you twist up enough, ~~cohomology~~ commutes with base change to every point.

Precisely: there exists M such that for every $m \geq M$, and every $p \in \operatorname{Spec} A$,

$$K(p) \otimes_A H^i(\mathbb{P}^n_A, \mathcal{F}(m)) \longrightarrow H^i(\mathbb{P}^n_p, \mathcal{F}|_{\mathbb{P}^n_p}(m))$$

is an isomorphism.

This statement is a bit misleading...

also $H^{>0} = 0$ in this range.

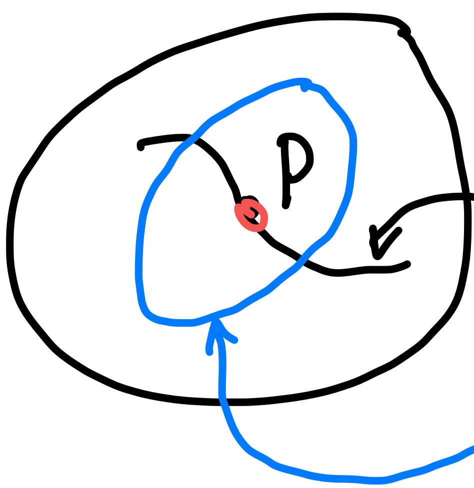
For $i > 0$, we get the "Spencer isomorphism" $0 \cong 0$.

Rk: universal vanishing.

Fourth step: get the (candidate) scheme structure on these (topological) strata.

Consider $\pi_* \mathcal{F}(M), \dots, \pi_* \mathcal{F}(M+n)$ on Spec A, which we used to find the strata.

Near a point p on one of these strata:



Spec A

stratum

set-th. intersection of (loc. closed) "flattening strata" for $\pi_* \mathcal{F}(M), \dots, \pi_* \mathcal{F}(M+n)$

Spec A_f

all closed (not just locally closed) in Spec A_f

In Spec A_f :

ideals: closed subscheme Spec$(A_f$ / ideal$)$

I_0 ~ flattening closed subscheme for $\pi_* \mathcal{F}(M)$

I_1 ~ flattening closed subscheme for $\pi_* \mathcal{F}(M+1)$

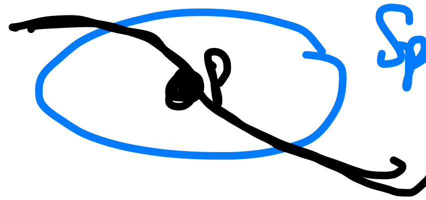
so $I_0 + I_1$ ~ biggest closed subscheme of Spec A_f where $\pi_* \mathcal{F}(M)$ and $\pi_* \mathcal{F}(M+1)$ are locally free of desired rank.

⋮

I_n ~ flattening closed subscheme for $\pi_* \mathcal{F}(M+n)$

$I_0 + \dots + I_n$ ~ biggest closed subscheme of Spec A_f where $\pi_* \mathcal{F}(M)$, ... $\pi_* \mathcal{F}(M+n)$ are locally free of desired rank.

Keep going!



Spec A_f

$I_{n+1}, I_{n+2}, \dots$ flattening subscheme for $\pi_*\mathcal{F}(M+n+1)$, $\pi_*\mathcal{F}(M+n+2), \dots$

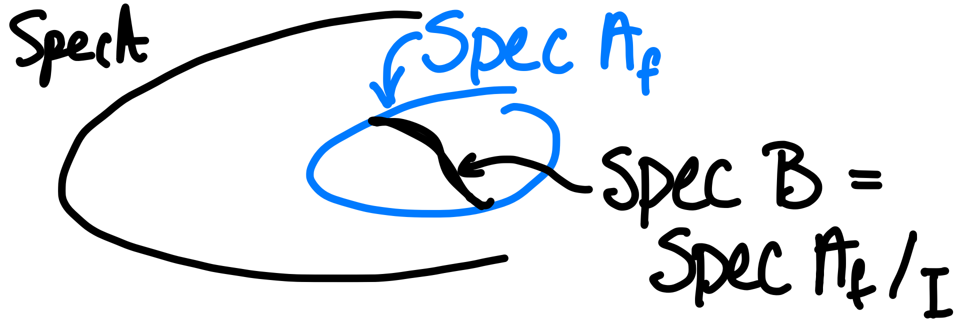
so consider

$$I_0+\dots+I_{n+1} \subseteq I_0+\dots+I_{n+2} \subseteq \dots$$

which eventually stabilizes (Noetherianity!), to I, say.

→ This will be our flattening stratification. ←

Recall



$$\begin{tikzcd} \mathcal{F}_B \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\ \mathbb{P}^n_B \arrow[r] \arrow[d, "\beta"'] & \mathbb{P}^n_A \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

Fifth Step $\quad I = I_0 + I_1 + I_2 + \cdots$

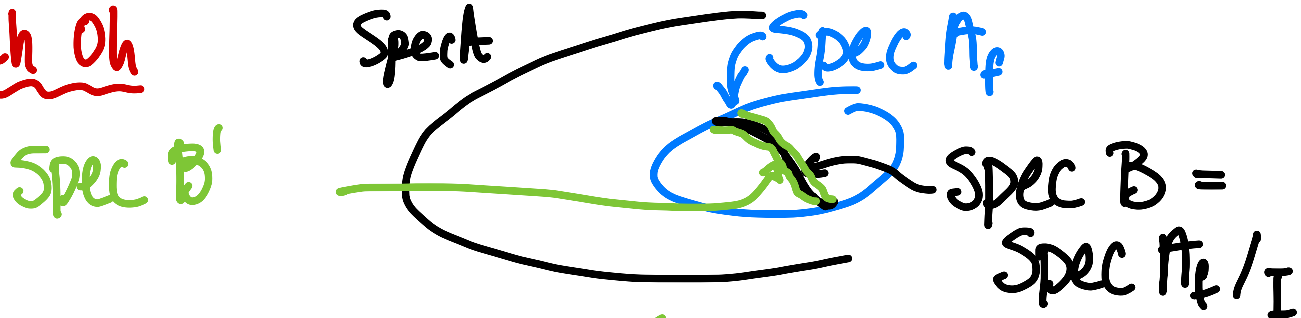
$\mathcal{F}$ is flat over this $\operatorname{Spec} \underbrace{A_f / I}_{B}$.

Reason: For this fixed Spec B, (higher) pushforwards of $\mathcal{F}(m)$ commute with base change for $m \geq M'$ for some M'.

"First Lemma"

Thus $\beta_* \mathcal{F}_B(m)$ is locally free for $m \gg 0$, so (by our criterion) $\mathcal{F}_B$ is flat!

Uh Oh



Fifth Step $\quad I' = I_1 + I_2 + \cdots$

$$\begin{tikzcd} & \mathcal{F}_{B'} \arrow[dl, dash] & & \mathcal{F} \arrow[dl, dash] \\ \mathbb{P}^n_{B'} \arrow[rr] \arrow[d, "\beta"'] & & \mathbb{P}^n_A \arrow[d] & \\ \operatorname{Spec} B' \arrow[rr] & & \operatorname{Spec} A & \end{tikzcd}$$

$\mathcal{F}$ is flat over this Spec $\overbrace{A_f/I'}^{B'}$.

Reason: For this *fixed* Spec B', (higher) pushforwards of $\mathcal{F}(m)$ commute with base change for $m \geq M'$ for some M'.

Thus $\beta_* \mathcal{F}_{B'}(m)$ is locally free for $m \gg 0$, so (by our criterion) $\mathcal{F}_{B'}$ is flat!

How to resolve this?

It must be true (although we haven't shown this) that

$$I_0 + I_1 + I_2 + \cdots = I_1 + I_2 + \cdots$$

If we can show this, then we will undoubtedly show:

$$\begin{aligned} I_0 + I_1 + I_2 + \cdots &= I_1 + I_2 + \cdots \\ &= I_2 + I_3 + \cdots \\ &\vdots \\ &= I_N + I_{N+1} + \cdots \end{aligned}$$

Ben will soon explain this.

Sixth step

This (on our neighborhood $\operatorname{Spec} A_f$ of $\mathfrak{p}$) satisfies our universal property!

Proof: One direction

$$\begin{tikzcd}
\mathcal{F}_Y \arrow[d, dash] & & \mathcal{F}_B \arrow[d, dash] & & \mathcal{F} \arrow[d, dash] \\
\mathbb{P}^n_Y \arrow[rr] \arrow[d] \arrow[drr, phantom, "\ulcorner", very near start] & & \mathbb{P}^n_B \arrow[rr] \arrow[d] \arrow[drr, phantom, "\ulcorner", very near start] & & \mathbb{P}^n_A \arrow[d] \\
Y \arrow[rr] & & \operatorname{Spec} B \arrow[rr] \arrow[d, phantom, "\shortparallel"] & & \operatorname{Spec} A_f \\
 & & A_f/I & &
\end{tikzcd}$$

Why is $\mathcal{F}_Y$ flat over Y?

Reason:

Sixth step (continued)

This (on our neighborhood $\operatorname{Spec} A_f$ of p) satisfies our universal property!

Proof: (continued) Other direction: Given

$$\begin{tikzcd}
\mathcal{F}_Y \arrow[d, dash] & & \mathcal{F}_{A_f} \arrow[d, dash] \\
\mathbb{P}^n_Y \arrow[d] \arrow[rr] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathbb{P}^n_{A_f} \arrow[d, dash] \\
Y \arrow[rr, "p"] & & \operatorname{Spec} A_f
\end{tikzcd}$$

with $\mathcal{F}_Y$ flat$/Y$ and the Hilbert polynomials of fibers $= p(t)$, why does $Y \to \operatorname{Spec} A_f$ factor $Y \to \operatorname{Spec} B \to \operatorname{Spec} A_f$?

We reduce to the case where Y is affine.

$M'' > M$

By our "First Lemma", for $M'' \gg 0$, $m \geq M''$ the higher pushforwards of $\mathcal{F}_{A_f}(m)$ and $\mathcal{F}_Y(m)$ are 0, and pushforward of $\mathcal{F}_{A_f}$ commutes with base change:

locally free

$$p^*(\pi_{A_f *})\mathcal{F}_{A_f}(m) \xrightarrow{\sim} (\pi_{Y *})\mathcal{F}_Y(m)$$

from last page

$$\boxed{\begin{tikzcd} \mathcal{F}_Y \arrow[d, dash] & & \mathcal{F}_{A_f} \arrow[d, dash] \\ \mathbb{P}^n_Y \arrow[rr] \arrow[d] \arrow[drr, phantom, "\ulcorner", very near start] & & \mathbb{P}^n_{A_f} \arrow[d, dash] \\ Y \arrow[rr, "e"] & & \operatorname{Spec} A_f \end{tikzcd}}$$

$$p^*(\pi_{A_f *})\mathcal{F}_{A_f}(m) \xrightarrow{\sim} (\pi_{Y *})\mathcal{F}_Y(m)$$

locally free

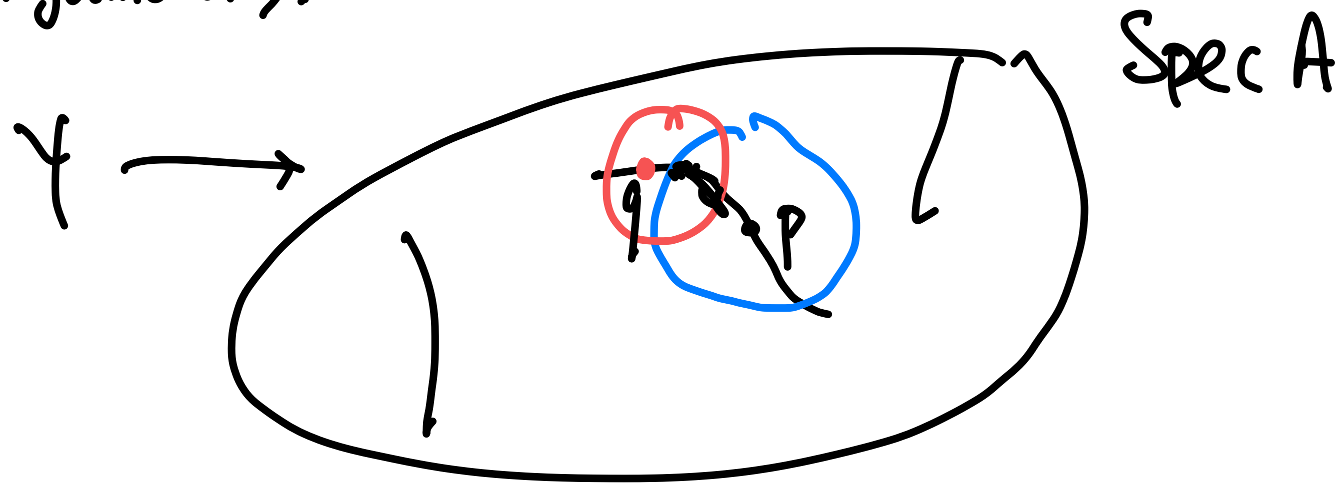
Then by the universal property of the flattenning stratification of $\mathcal{F}_Y(m)$, $Y \to \operatorname{Spec} A_f$ factors through $\operatorname{Spec} A_f / I_{m-M}$, for all $m \geq M_1''$, so it factors through $\operatorname{Spec} A_f / I_{M''-M} + I_{M''-M+1} + \cdots$

By Ben-from-the-future: $I_{M''-M} + I_{M''-M+1} + \cdots = I_0 + I_1 + \cdots$

Seventh step

(that you may not have noticed): all of these constructions glue together (by a universal property argument).

fix Hilb pol.
$p(t)$



This completes the proof! (Do you agree?)

//

So now Ben will explain his argument.

the Quot scheme

The Quot FUNCTOR

Families of (flat) quotients of $\mathcal{O}_{\mathbb{P}^n}^{\oplus p}$: here is the contravariant FUNCTOR from (Schemes) to (Sets) that we'll see is representable:

Quot: $B \rightsquigarrow$

$$\begin{tikzcd} \mathcal{O}^{\oplus p} \arrow[r, two heads] \arrow[d, dash] & \mathcal{F} \\ \mathbb{P}^n_B \arrow[d] & \\ B & \end{tikzcd}$$

$\mathcal{F}$ flat over B

(depends on n, p)

Observations:

(i) If you wish:

$$B \rightsquigarrow \begin{array}{c} \text{flat over } B \\ 0 \to \mathcal{E} \to \mathcal{O}^{\oplus p} \to \mathcal{F} \to 0 \\ | \\ \mathbb{P}^n_B \\ \downarrow \\ B \end{array}$$

(ii) Special cases:

$p = 1$: Hilbert scheme

$n = 0$: Grassmannian

flat over B

$$B \rightsquigarrow \begin{array}{c} 0 \to \mathcal{E} \to \mathcal{O}^{\oplus p} \to \mathcal{F} \to 0 \\ | \\ \mathbb{P}^n_B \\ \downarrow \\ B \end{array}$$

(iii) breaks up into subFUNCTORs that are simultaneously open and closed, indexed by

the Hilbert polynomial $\chi(\mathbb{P}^n, \mathcal{F}(t))$ of $\mathcal{F}$

or, equivalently,

the Hilbert polynomial $\chi(\mathbb{P}^n, \mathcal{E}(t))$ of $\mathcal{E}$

Reality check: what is the Hilbert polynomial in the case $n=0$,

$$0 \to \mathcal{E} \to \mathcal{O}^{\oplus p} \to \underbrace{\mathcal{F}}_{\text{rank } k} \to 0$$

i.e., the Grassmannian?

For fixed $n \in \mathbb{Z}^{\geq 0}$, $\rho \in \mathbb{Z}^{>0}$, $p(t) \in \mathbb{Q}[t]$

Think: $\mathbb{P}^n$ $\quad$ $\mathcal{O}^{\oplus \rho}$ $\quad$ Hilbert polynomial

Define the corresponding Quot FUNCTOR.

Theorem (Grothendieck) This functor is representable. (This is the definition of the Quot scheme for n, ρ, $p(t)$.)

The Quot scheme is projective over $\operatorname{Spec} \mathbb{Z}$.

This is our medium-term goal.

Let's think about how we would prove this.

Over a field k, suppose we have an object (of Quot(Spec k)):

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O}^{\oplus p} \arrow[r] \arrow[d, dash] & \mathcal{F} \arrow[r] & 0 \\ & & \mathbb{P}^n_k & & \end{tikzcd}$$

where $\mathcal{E}$ and $\mathcal{F}$ have known Hilbert polynomial. Thanks to Mumford*, we know there is some M (depending only on n, p, $p(t)$) such that $\mathcal{E}$ and $\mathcal{F}$ are M-regular. How will we use this?

* last week, I think

Hence for any family in $\mathrm{Quot}^{n,p,p(t)}(B)$,

$$\begin{array}{c} 0 \to \mathcal{E} \to \mathcal{O}^{\oplus p} \to \mathcal{F} \to 0 \\ | \\ \mathbb{P}^n_B \\ \downarrow \\ B \end{array}$$

for $m \geq M$, we feel like twisting by m:

$$\begin{array}{c} 0 \to \mathcal{E}(m) \to \mathcal{O}(m)^{\oplus p} \to \mathcal{F}(m) \to 0 \\ | \\ \mathbb{P}^n \end{array}$$

Then what?

For each point $q \in B$,

$$H^{i>0}(\mathbb{P}^n, \mathcal{E}(m)) = H^{i>0}(\mathbb{P}^n, \mathcal{O}(m)^{\oplus p})$$
$$= H^{i>0}(\mathbb{P}^n, \mathcal{F}(m)) = 0$$

$\pi_* \mathcal{F}(m)$ is a locally free sheaf of rank $p(m)$.

We have an exact sequence of locally free sheaves on B.

$$0 \to \pi_* \mathcal{E}(m) \to (\pi_* \mathcal{O}(m))^{\oplus p} \to \pi_* \mathcal{F}(m) \to 0$$

B

Why?

For each point $q \in B$,

$$H^{i>0}(\mathbb{P}^n, \mathcal{E}(m)) = H^{i>0}(\mathbb{P}^n, \mathcal{O}(m)^{\oplus \ell})$$
$$= H^{i>0}(\mathbb{P}^n, \mathcal{F}(m)) = 0$$

$\pi_* \mathcal{F}(m)$ is a locally free sheaf of rank $p(m)$.

We have an exact sequence of locally free sheaves on B.

$$0 \to \pi_* \mathcal{E}(m) \to (\pi_* \mathcal{O}(m))^{\oplus \ell} \to \pi_* \mathcal{F}(m) \to 0$$

This gives a map $B \to \underbrace{G\left(p_{\mathcal{E}}(m), \binom{m+\ell}{m}, p_{\mathcal{F}}(m)\right)}_{\text{(explain)}}$

Given

$$0 \to \pi_* \mathcal{E}(m) \to \left(\pi_* \mathcal{O}(m)\right)^{\oplus \rho} \to \pi_* \mathcal{F}(m) \to 0 \quad (*)$$

(i.e. $0 \to V \to \pi_* \mathcal{O}(m)^{\oplus \rho} \to W \to 0$, along with the knowledge that it is isomorphi to some $(*)$),

I claim we can recover $\mathcal{F}$ from this information.

$\therefore$ Quot is a subFUNCTOR of the Grassmannian

Given

$$0 \to \pi_* \mathcal{E}(m) \to \left(\pi_* \mathcal{O}(m)\right)^{\oplus P} \to \pi_* \mathcal{F}(m) \to 0 \quad (*)$$

(i.e. $0 \to V \to \pi_* \mathcal{O}(m)^{\oplus P} \to W \to 0$, along with the knowledge that it is isomorphic to some $(*)$),

I claim we can recover $\mathcal{F}$ from this information.

$\therefore$ Quot is a subFUNCTOR of the Grassmannian

I also claim it is a locally closed subFUNCTOR of the Grassmannian.

We have proved:

<u>Theorem (Grothendieck)</u> This functor is representable. (This is the definition of the Quot scheme for n, ρ, $p(t)$.)

The Quot scheme is projective over $\operatorname{Spec} \mathbb{Z}$.

quasi

It is a locally closed subscheme of a Grassmannian!

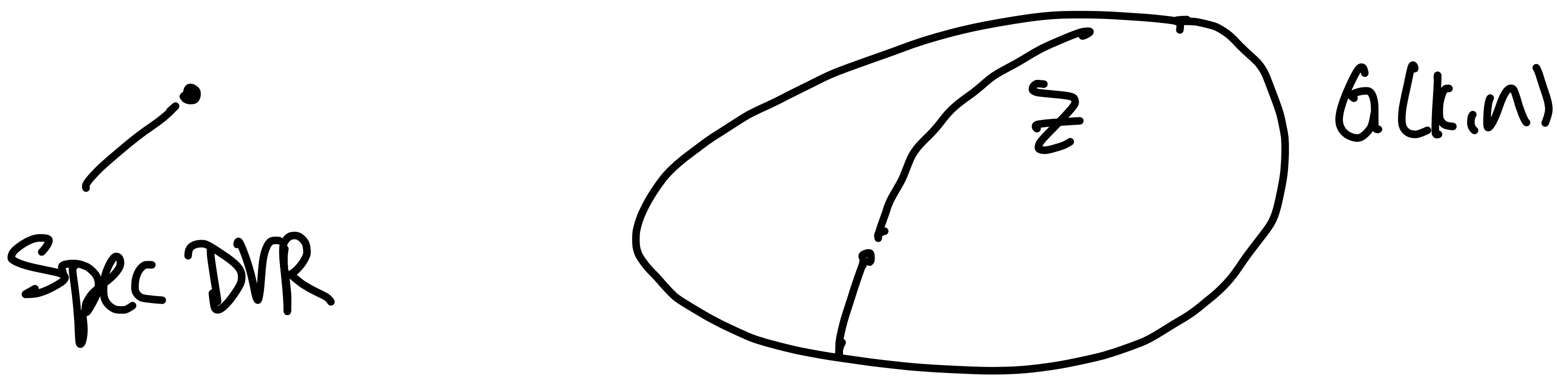
To complete our proof, we want to show that this locally closed subscheme of the Grassmannian is in fact a closed subscheme.

Someone trying to impress you: let's use the valuative criterion of properness! (Then you have to prove the valuative criterion of properness.)

Cheaper by far: To show that a locally closed subscheme is closed, you need only show that the underlying locally closed set is closed.

The ambient space is $G(k,n)$ over Spec $\mathbb{Z}$.

I present to you: the valuative criterion to check if a locally closed subset Z of $G(k,n)$ is closed.



$$\begin{tikzcd} \operatorname{Spec} K(A) \arrow[r] \arrow[d] & Z \arrow[d, hook] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists ?"] & G(k,n) \end{tikzcd}$$

(ok, it is the same as the valuative criterion of properness)

So we now have a question.

Given a DVR A, and:

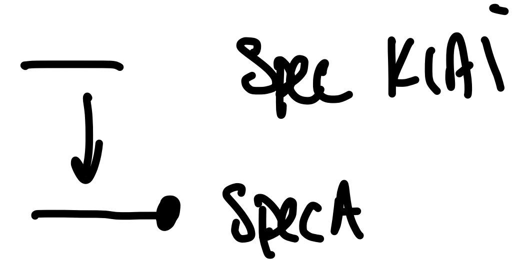
$$\begin{array}{ccc} \mathcal{O}^{\oplus p} & \longrightarrow & \mathcal{F} \\ \downarrow & & \\ \mathbb{P}^n_{K(A)} & & \end{array}$$

Can you extend this to:

$$\begin{array}{ccc} \mathcal{O}^{\oplus p} & \longrightarrow & \mathcal{F}_* \text{ flat} \\ \downarrow & & \\ \mathbb{P}^n_A & & \end{array}$$

?

Grothendieck says yes. Why is he right?



$i: \operatorname{Spec} K(A) \longrightarrow \operatorname{Spec} A$ is an affine morphism

$$0 \to \mathcal{E}_{K(A)} \to \mathcal{O}_{K(A)}^{\oplus \ell} \longrightarrow \mathcal{F}_{K(A)} \to 0$$

coherent

$$\underbrace{\mathcal{O}_{\mathbb{P}^n_A}^{\oplus \ell}}_{\text{coherent}} \longrightarrow i_* \mathcal{O}_{\mathbb{P}^n_{K(A)}}^{\oplus \ell} \twoheadrightarrow \underbrace{i_* \mathcal{F}_{K(A)}}_{\text{quasicoherent}}$$

Let $\mathcal{F}_A$ be the image of $\mathcal{O}_{\mathbb{P}^n_A}^{\oplus \ell}$ in $i_* \mathcal{F}_{K(A)}$

coherent!

This extends the map over $K(A)$.

Furthermore: it is flat. Reason: flat = torsionfree.