Moduli Spaces in Algebraic Geometry

Math 245 A (Winter 2022)

Feb. 4, 2022.

Cohomology and Base Change

The Situation

$$\begin{tikzcd} W \arrow[r, "\psi'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \\ Z \arrow[r, "\psi"'] & Y \end{tikzcd}$$

$, \mathcal{F}$ coherent

π proper

(everything Noetherian)

$$\phi^p : \underset{F}{\psi^*} \underbrace{R^p \pi_*}_{H} \mathcal{F} \longrightarrow \underbrace{R^p (\pi')_*}_{H} \underset{F}{(\psi')^*} \mathcal{F}$$

When is this an isomorphism?

Translation: When does cohomology commute with base change?

(higher pushforward)

The Mumford Complex (last time)

Situation:

$$\begin{tikzcd} X \arrow[d, "\pi"'] & \mathcal{F} \arrow[l, dash] \\ Y = \operatorname{Spec} B & \end{tikzcd}$$

$\mathcal{F}$ coherent; flat$/Y$

π proper

B Noetherian

so $R^p\pi_*\mathcal{F} = \widetilde{H^p(X,\mathcal{F})}$ is coherent on B

Then there is a complex of finitely-generated free B-modules

$$\cdots \to B^{\oplus ?} \to \cdots \to B^{\oplus ?} \to B^{\oplus ?} \to B^{\oplus ?} \to 0$$

degree $\cdots$ $n-1$ n

that universally (i.e., after any base change) gives the cohomology of $\mathcal{F}$.

You want to understand $R^p\pi_* \mathcal{F}$, and how it behaves under base change? You need only consider

$$B^{\oplus \text{finite}} \longrightarrow \underset{\text{p^{th} step}}{B^{\oplus \text{finite}}} \longrightarrow B^{\oplus \text{finite}}$$

(two matrices with entries in B, that multiply to zero)

(How hard could that be?)

Grauert's Theorem:

Situation: $\begin{tikzcd} \mathcal{F} \arrow[d, dash] & \\ X \arrow[d, "\pi"] & \\ Y \end{tikzcd}$ $\mathcal{F}$ coherent; flat over Y; π proper; Y Noetherian

If $q \mapsto h^p(X_q, \mathcal{F}|_{X_q})$ is a locally constant function, and Y is reduced, then $R^p\pi_*\mathcal{F}$ is locally free (of that rank), and cohomology commutes with <u>any</u> base change of Y: ϕ^p is <u>always</u> an isomorphism

$$\begin{tikzcd} & & \mathcal{F} \arrow[dl, dash] \\ W \arrow[r, "\psi'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\pi"] & \\ Z \arrow[r, "\psi"'] & Y & \end{tikzcd}$$

$$\phi^p : \psi^* R^p\pi_*\mathcal{F} \xrightarrow{\sim} R^p(\pi')_*(\psi')^*\mathcal{F}$$

THE Cohomology and Base Change Theorem

Situation:

$$\begin{tikzcd} X_q \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ q \arrow[r, "\psi"] & Y \end{tikzcd}$$

$\mathcal{F}$ coherent; flat over Y (on X)

π proper

q point

Y Noetherian

If $\phi^p_q : (R^p\pi_*\mathcal{F})|_q \to H^p(X_q, \mathcal{F}|_{X_q})$ is surjective, then:

(i) There is some neighborhood $\mathcal{U}$ of q in Y so that cohomology commutes with <u>any</u> base change to $\mathcal{U}$. (In particular, ϕ^p_q is an isomorphism.)

(ii) Furthermore, ϕ^{p-1}_q is surjective if and only if $R^p\pi_*\mathcal{F}$ is locally free in some open neighborhood of q.

We are proving these following Eric Larson (2020).

We begin with some easy (but, I now believe, important) observations about maps of finite rank locally free sheaves $\mathcal{E} \xrightarrow{f} \mathcal{F}$ on a scheme X

Less Important definition

We say f is weakly of constant rank a if for every point $p \in X$, $f|_p : \mathcal{E}|_p \to \mathcal{F}|_p$ has rank a.

Important Definition

We say f is **strongly of constant rank a** if for every point $p \in X$, there is an open neighborhood U of p with

$$\begin{tikzcd}
\mathcal{E}|_U \arrow[rr, "f|_U"] \arrow[d, "\sim"] & & \mathcal{F}|_U \arrow[d, "\sim"] \\
\mathcal{O}_U^{\oplus(a+b)} \arrow[rr] \arrow[dr] & & \mathcal{O}_U^{\oplus(a+c)} \\
& \mathcal{O}_U^{\oplus a} \arrow[ur] &
\end{tikzcd}$$

projection from first a factors

inclusion as first a factors

<u>Note:</u> "strongly of constant rank":

- implies weakly of constant rank
- preserved by base change
- image, kernel, cokernel locally free; and their construction commutes with base change. Because locally we can write:

On U:

$$\begin{tikzcd}
0 \arrow[r] & \ker f \arrow[r] \arrow[d, leftrightarrow, "\sim"] & \mathcal{E} \arrow[r, "f"] \arrow[d, leftrightarrow, "\sim"] & \mathcal{F} \arrow[r] \arrow[d, leftrightarrow, "\sim"] & \operatorname{coker} f \arrow[r] \arrow[d, leftrightarrow, "\sim"] & 0 \\
0 \arrow[r] & \mathcal{O}^{\oplus b} \arrow[r] & \mathcal{O}^{\oplus (a+b)} \arrow[r] & \mathcal{O}^{\oplus (a+c)} \arrow[r] & \mathcal{O}^{\oplus c} \arrow[r] & 0
\end{tikzcd}$$

We'll see different ways of showing f is strongly of constant rank.

First criterion...

Lemma coker f is locally free iff f is strongly of constant rank

Proof As already stated, if f is strongly of constant rank a, then coker f is cheaply locally free — in the definition, the cokernel is locally isomorphic to $\mathcal{O}^{\oplus c}$.

Conversely, suppose coker f is locally free. We show it is strongly of constant rank near a given point p. Pick an affine open neighborhood Spec A of p over which $\mathcal{E}$, $\mathcal{F}$, and coker f are free of rank $a+b$, $a+c$, and c respectively.

$$\mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \operatorname{coker} f \longrightarrow 0$$

$$\begin{tikzcd}
A^{\oplus(a+b)} \arrow[r, "f"] & A^{\oplus(a+c)} \arrow[r, "g"] & A^{\oplus c} \arrow[r] \arrow[l, bend left, "g'"] & 0
\end{tikzcd}$$

Choose a splitting g' of g

Use this to give a <u>different</u> basis of $A^{\oplus(a+c)}$ so that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker g & \longrightarrow & A^{\oplus(a+c)} & \longrightarrow & A^{\oplus c} & \longrightarrow & 0 \\
 & & \wr & & & & & & \\
0 & \longrightarrow & A^{\oplus a} & \longrightarrow & A^{\oplus(a+c)} & \longrightarrow & A^{\oplus c} & \longrightarrow & 0
\end{array}$$

inclusion as first a factors

project to last c factors

$0 \to \ker f \to A^{\oplus(a+b)} \to A^{\oplus(c+c)} \to A^{\oplus c} \to 0$

Then we have:

$$0 \to \ker f \longrightarrow A^{\oplus(a+b)} \longrightarrow A^{\oplus a} \longrightarrow 0$$

Do the same thing here, so

$$0 \to A^{\oplus b} \longrightarrow A^{\oplus(a+b)} \longrightarrow A^{\oplus a} \longrightarrow 0$$

inclusion of first b factors

project to last a factors.

Second Criterion:

Corollary If X is reduced, weakly of constant rank
$=$ strongly of constant rank

Proof: weakly of constant rank

$\Rightarrow$ coker f (a finitely presented sheaf, with presentation $\mathcal{E} \xrightarrow{f} \mathcal{F} \to \operatorname{coker} f \to 0$ (*))

has constant rank (as (*) remains exact upon applying $\otimes k(p)$ for any $p \in X$).

But finitely presented sheaves of constant rank on reduced schemes are locally free. $\square$

Proof of Grauert's Theorem:

$$\begin{tikzcd} X_q \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \arrow[r, phantom, "\mathcal{F}"] & \\ \text{point } q \arrow[r, "\varphi"] & Y \end{tikzcd}$$

$\mathcal{F}$ coherent; flat over Y

π proper

Y Noetherian

(If $q \mapsto h^p(X_q, \mathcal{F}|_{X_q})$ is a locally constant function, and Y is reduced, then $R^p\pi_*\mathcal{F}$ is locally free (of that rank), and cohomology commutes with any base change of Y: ϕ^p is always an isomorphism.)

Mumford Complex at q $\quad K^{p-1} \xrightarrow{\alpha} K^p \xrightarrow{\beta} K^{p+1}$

$$h^p(X_q, \mathcal{F}|_{X_q}) = \dim K^p - \operatorname{rank}_q \alpha - \operatorname{rank}_q \beta$$

$\rightarrow \alpha, \beta$ ~~weakly~~ strongly of constant rank.

locally free

$$0 \to \ker\alpha \to K^{p-1} \xrightarrow{\alpha} K^{p} \to \operatorname{coker}\alpha \to 0$$

$$0 \to \ker\beta \to K^{p} \xrightarrow{\beta} K^{p+1} \to \operatorname{coker}\beta \to 0$$

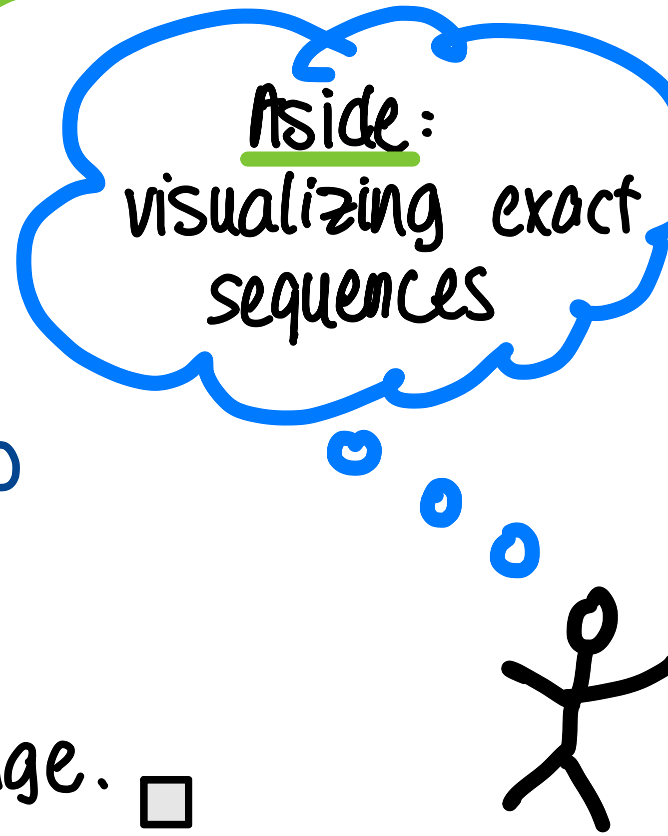
Then $0 \to \operatorname{Im}\alpha \to \operatorname{Ker}\beta \to H^p \to 0$

No, better:

$$0 \to H^p \to \operatorname{Coker}\alpha \to \operatorname{Im}\beta \to 0$$

Thus H^p is flat.

Everything commutes with base change. $\square$



ASIDE

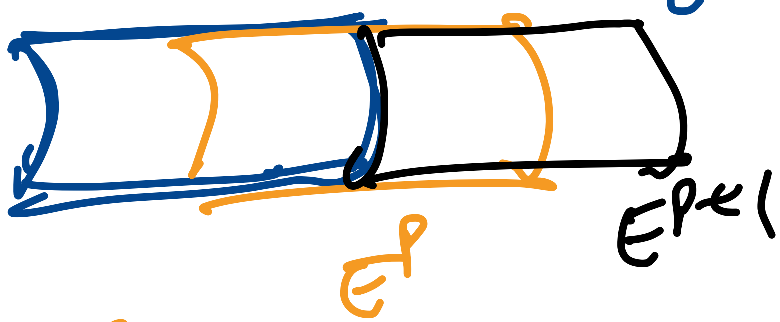
vector space V

$\boxed{\Phi: V \to W.}$

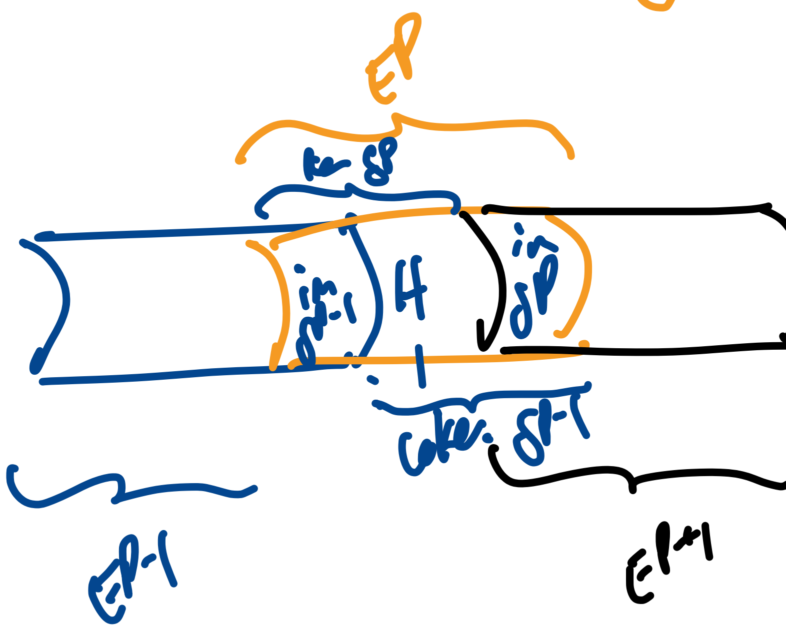
$V \to W$

V

$E^{p-1} \to E^{p} \to E^{p+1}$



$E^{p-1} \xrightarrow{\delta^{p-1}} E^{p} \xrightarrow{\delta^{p}} E^{p+1}$



$\mathcal{E} \xrightarrow{f} \mathcal{F}$

Third Criterion:

Lemma Fix $q \in X$. $\ker(f)|_q \to \ker(f|_q)$ is surjective if and only if f is strongly of constant rank in some neighborhood of q.

Proof Again, one direction, we have discussed already: if f is strongly of constant rank, then near q we can write it

$$0 \to \underset{\substack{\| \\ \ker f}}{\mathcal{O}^{\oplus b}} \to \mathcal{O}^{\oplus (a+b)} \xrightarrow{f} \mathcal{O}^{\oplus (a+c)} \to \underset{\substack{\| \\ \operatorname{coker} f}}{\mathcal{O}^{\oplus c}} \to 0$$

Then this remains exact upon restriction to q (exact $\otimes$ $k(\cancel{p})$ anything is exact). So $(\ker f)|_q \cong \ker(f|_q)$.

How about the other direction?

We always have a map

$$(\ker f)|_q \to \ker(f|_q)$$

In an affine neighborhood:

$$A^{\oplus(a+b)} \to A^{\oplus(a+c)}$$

Restrict to a point

$$K(q)^{\oplus(a+b)} \to K(q)^{\oplus(a+c)}$$

Choose basis of kernel $\bar{u}_1, \dots, \bar{u}_b$ of left side.

Extend it to a basis $\bar{u}_1, \dots, \bar{u}_b, \bar{v}_1, \dots, \bar{v}_a$ of left side.

FHHF point of view:

$$0 \to \mathcal{E} \xrightarrow{f} \mathcal{F} \to 0$$

$$F = \otimes K(p)$$

$$FH \to HF$$

$$(\ker f)|_p \to \ker(f_p)$$

Lift these to $u_1, \dots, u_b, v_1, \dots, v_a \in A^{\oplus(a+b)}$ so that $u_1, \dots, u_b \in \ker f$.

Here we use the surjectivity $(\ker f)|_{\mathfrak{p}} \to \ker(f|_{\mathfrak{p}})$ as well as Nakayama's lemmma for $A^{\oplus(a+b)}$. With this new basis for $A^{\oplus(a+b)}$, we have:

$$\begin{aligned} A^{\oplus(a+b)} &\longrightarrow A^{\oplus(a+c)} \\ u_1, \dots, u_b &\longmapsto 0 \end{aligned}$$

leaving $0 \to A^{\oplus a} \hookrightarrow A^{\oplus(a+c)} \xrightarrow{g} \operatorname{coker} f \to 0$

Once again, but for g: $(\ker g)|_p \twoheadrightarrow \ker(g|_p)$.

So by the same argument, we can choose a new basis for $A^{\oplus(a+c)}$ so that $A^{\oplus a}$ maps isomorphically onto the first a summands. $\square$

Lemma Suppose

$$\begin{tikzcd} K^{p-1} \arrow[r, "\delta_K^{p-1}"] \arrow[d] & K^p \arrow[r, "\delta_K^{p-1}"] \arrow[d, "?"] & K^{p+1} \arrow[d, "?"] \\ J^{p-1} \arrow[r, "\delta_J^{p-1}"'] & J^p \arrow[r, "\delta_J^p"'] & J^{p+1} \end{tikzcd}$$

is a map of complexes, termwise surjective.

Then $H^p(K^\bullet) \to H^p(J^\bullet)$ is surjective if and only if $\ker \delta_K^p \to \ker \delta_J^p$ is surjective.

Proof Consider:

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{im} \delta_K^{p-1} \arrow[r] \arrow[d] & \ker \delta_K^p \arrow[r] \arrow[d] & H^p(K^\bullet) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{im} \delta_J^{p-1} \arrow[r] & \ker \delta_J^p \arrow[r] & H^p(J^\bullet) \arrow[r] & 0 \end{tikzcd}$$

$\square$

Proof of: THE Cohomology and Base Change Theorem

Situation:

$$\begin{tikzcd} X_q \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ q \arrow[r, "\varphi"] & Y \end{tikzcd}$$

$\mathcal{F}$ coherent; flat over Y

π proper

point q

Y Noetherian

If $\phi_q^p : (R^p\pi_*\mathcal{F})|_q \to H^p(X_q, \mathcal{F}|_{X_q})$ is surjective, then:

(i) There is some neighborhood U of q in Y so that cohomology commutes with <u>any</u> base change to U. (In particular, ϕ_q^p is an isomorphism.)

(ii) Furthermore, ϕ_q^{p-1} is surjective if and only if $R^p\pi_*\mathcal{F}$ is locally free in some open neighborhood of q.

Mumford complex:

near point
$$\begin{tikzcd} K^{p-1} \arrow[r] \arrow[d] & K^{p} \arrow[r, "\delta^p"] \arrow[d] & K^{p+1} \arrow[d] \\ K^{p-1} \otimes_B k(q) \arrow[r] & K^{p} \otimes_B k(q) \arrow[r, "\delta^p \otimes k"] & K^{p} \otimes_B k(q) \end{tikzcd}$$
free B-modules

at point

Hypothesis:

$\phi^p_q : (R^p \pi_* \mathcal{F})|_q \to H^p(X_q, \mathcal{F}|_{X_q})$ is surjective,

last lower

iff $(\ker \delta^p) \otimes k \longrightarrow \ker(\delta^p \otimes k)$ is surjective

iff $\boxed{\delta^p \text{ strongly constant rank. near point}}$

$\Longrightarrow$ $\ker \delta^p$ commutes with any base change near point.

Consider $K^{p-1} \to \ker \delta^p \to H^p(K^\bullet) \to 0$

$\therefore$ $H^p(K^\bullet)$ commutes with any base change.

Still to prove:

(ii) Furthermore, ϕ_q^{p-1} is surjective if and only if $R^p\pi_*\mathcal{F}$ is locally free in some open neighborhood of q.

consider again $K^{p-1} \to \ker \delta^p$ cokernel is $H^p(K^\cdot)$

locally free

Now $H^p(K^\cdot)$ locally free iff $K^{p-1} \xrightarrow{\alpha} \ker \delta^p$ is strongly of constant rank iff (since δ^p is strongly of constant rank) δ^{p-1} is strongly of constant rank iff

$$H^{p-1}(K^\cdot) \to H^{p-1}(K^\cdot|_x) \text{ is surjective.}$$

$K^{p-1} \xrightarrow{\delta^{p-1}} K^p \to \operatorname{im}\alpha \to 0$

$\square$