Moduli Spaces in Algebraic Geometry

Math 245A (Winter 2022)

Feb. 2, 2022.

Cohomology and Base Change

The Situation

$$\begin{tikzcd} & & \mathcal{F} \\ W \arrow[r, "\psi'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \arrow[ur, phantom, "\ni"] & \\ Z \arrow[r, "\psi"'] & Y & \end{tikzcd}$$

coherent

proper

(everything Noetherian)

$$\phi^p : \underbrace{\psi^*}_{F} \underbrace{R^p \pi_*}_{H} \mathcal{F} \longrightarrow \underbrace{R^p (\pi')_*}_{H} \underbrace{(\psi')^*}_{F} \mathcal{F}$$

When is this an isomorphism?

(higher pushforward)

Translation: When does cohomology commute with base change?

$$\begin{tikzcd} W \arrow[r, "\psi'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] & \mathcal{F} \arrow[l, dash] \\ Z \arrow[r, "\psi"'] & Y & \end{tikzcd}$$

$$\phi^p : \psi^* R^p \pi_* \mathcal{F} \xrightarrow{\sim ?} R^p(\pi')_* (\psi')^* \mathcal{F}$$

For example,

we have already seen:

If $X \to Y$ is projective, then replacing $\mathcal{F}$ by $\mathcal{F}(M)$ for $M \gg 0$, the answer is "yes".

(our "first lemma")

If ψ is flat, the answer is yes.

If $\mathcal{F}$ is flat and $R^{i>0}\pi_*\mathcal{F} = 0$, the answer is yes.

$$\begin{tikzcd} W \arrow[r, "\psi'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] & \mathcal{F} \arrow[l, dash] \\ Z \arrow[r, "\psi"'] & Y & \end{tikzcd}$$

$$\phi^p : \psi^* R^p \pi_* \mathcal{F} \xrightarrow{\sim ?} R^p (\pi')_* (\psi')^* \mathcal{F}$$

We will see (even if we've seen them before):

If $\mathcal{F}$ is flat / Y,

- the Semicontinuity Theorem: $q \in Y \mapsto h^p(X_q, \mathcal{F}|X_q)$ is an upper semicontinuous function on Y

- Euler characteristic $\chi(X_q, \mathcal{F}|X_q)$ is a locally constant function on Y.

And more...

Situation: $\begin{tikzcd} \mathcal{F} \arrow[d, dash] & \\ X \arrow[d] \\ Y \end{tikzcd}$ coherent; flat over Y

proper

Noetherian

Grauert's Theorem: If $q \mapsto h^p(X_q, \mathcal{F}|_{X_q})$ is a locally constant function, and Y is reduced, then $R^p\pi_*\mathcal{F}$ is ~~a vector bundle~~ locally free (of that rank), and cohomology commutes with any base change of Y:

ϕ^p is always an isomorphism

$$\begin{tikzcd} W \arrow[r, "\psi'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] & \mathcal{F} \arrow[l, dash] \\ Z \arrow[r, "\psi"'] & Y \end{tikzcd}$$

$$\phi^p : \psi^* R^p\pi_*\mathcal{F} \xrightarrow{\overset{?}{\sim}} R^p(\pi')_*(\psi')^*\mathcal{F}$$

Most amazing: $\phi_q^p : (R^p\pi_*\mathcal{F})|_q \to H^p(X_q, \mathcal{F}|_{X_q})$

$\mathcal{F}$ coherent; flat over Y

$$\begin{tikzcd} X_q \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ q \arrow[r, "\psi"] & Y \end{tikzcd}$$

π proper

point q ; Y Noetherian

THE Cohomology and Base Change Theorem

Suppose ϕ_q^p is surjective. Then:

(i) There is some neighborhood U of q in Y so that cohomology commutes with <u>any</u> base change to U. (In particular, ϕ_q^p is an isomorphism.)

Most amazing: $\phi^p : (R^p\pi_*\mathcal{F})|_q \to H^p(X_q, \mathcal{F}|_{X_q})$

$$\begin{tikzcd} & & \mathcal{F} \arrow[dl, dash] & \text{coherent; flat over } Y \\ & X_q \arrow[r] \arrow[d] & X \arrow[d, "\pi"] & \text{proper} \\ \text{point} & q \arrow[r, "\psi"] & Y & \text{Noetherian} \end{tikzcd}$$

THE Cohomology and Base Change Theorem

Suppose ϕ^p_q is surjective. Then:

(i) There is some neighborhood U of q in Y so that cohomology commutes with any base change to U. (In particular, ϕ^p_q is an isomorphism.)

(ii) Furthermore, ϕ^{p-1}_q is surjective if and only if $R^p\pi_*\mathcal{F}$ is locally free in some open neighborhood of q.

equivalently, $(R^p\pi_*\mathcal{F})_q$ is a free $\mathcal{O}_{Y,q}$-module

implies $h^p(X_{q'}, \mathcal{F}|_{X_{q'}})$ is constant for q' near q

no reduced hypotheses!

can then use this for $p-1$...

This is super useful.

Example, relevant to us: moduli of degree d hypersurfaces in $\mathbb{P}^n$.

Suppose
$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\text{flat}"'] & \mathbb{P}^n_B \\ B & \end{tikzcd}$$
is a flat family, whose fibers over points of B are degree d hypersurfaces. We expect to produce from this $B \to \mathbb{P}^{\binom{n+d}{d}-1}$. How?

Answer: Consider

$$\begin{tikzcd} 0 \arrow[r] & I_X(d) \arrow[r] & \mathcal{O}_{\mathbb{P}^n_B}(d) \arrow[r] \arrow[d, phantom, "\mid"] & \mathcal{O}_X(d) \arrow[r] & 0 \\ & & \mathbb{P}^n_B \arrow[d] & & \\ & & B & & \end{tikzcd}$$

Notice: flat over B:

Also, for every point q of B,

$$H^{i>0}(\mathbb{P}^n_q, I_{X_q}(d)) = H^{i>0}(\mathbb{P}^n_q, \mathcal{O}_{X_q}(d)) = 0$$

$$h^0(\mathbb{P}^n_q, I_{X_q}(d)) = 1, \qquad h^0(\mathbb{P}^n_q, \mathcal{O}_{\mathbb{P}^n_q}(d)) = \binom{n+d}{d},$$

$$h^0(\mathbb{P}^n_q, \mathcal{O}_{X_q}(d)) = \binom{n+d}{d} - 1$$

(This is a question about hypersurfaces over a field.)

Hence: $R^{i>0}\pi_* \mathcal{I}_X(d) = R^{i>0}\pi_* \mathcal{O}_X(d) = 0$

by Cohomology and Base Change, and

$R^{i>0}\pi_* \mathcal{O}_{\mathbb{P}^n}(d) = 0$ by CBC or direct calculation.

Then $\pi_* \mathcal{I}_X(d)$ is locally free of rank 1

$\pi_* \mathcal{O}_{\mathbb{P}^n}(d)$ is free of rank $\binom{n+d}{d}$

and $\pi_* \mathcal{O}_X(d)$ is locally free of rank $\binom{n+d}{d} - 1$.

rank 1 $\qquad$ free $\qquad$ rank $\binom{n+d}{d}-1$

We thus have:

$$0 \to \pi_* I_X(d) \longrightarrow \mathcal{O}^{\oplus \binom{n+d}{d}} \to \pi_* \mathcal{O}_X(d) \longrightarrow 0$$

$$\begin{array}{c} | \\ B \end{array}$$

which is the same as a map $B \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$!

Why is this "reversible"? (Discuss)

(if you want).

To prove these Theorems, we work much less than you might expect!

Situation:

$$\begin{array}{ll} \mathcal{F} & \text{flat}/_Y, \text{ coherent} \\ X & \\ \downarrow & \text{proper} \\ Y = \operatorname{Spec} B & \text{Noetherian} \end{array}$$

Key Idea (take 1)

We'll describe a complex of free, finite rank, B-modules:

$$\begin{array}{ccccccccccc} & & n-2 & & n-1 & & n & & & & \\ \cdots & \to & B^{\oplus ?} & \to & B^{\oplus ?} & \to & B^{\oplus ?} & \to & 0 & \to & 0 \to \cdots \end{array}$$

$$\rightsquigarrow \quad \cdots \to K^{n-2} \to K^{n-1} \to K^n \to$$

matrix
$\boxed{}$ entries cal?

It will compute the cohomology of $\mathcal{F}$

e.g. $\quad H^p(X,\mathcal{F}) = \text{cohomology}\left(K^{p-1} \to K^p \to K^{p+1}\right)$

$R^p\pi_*\mathcal{F} = \widetilde{H^p(X,\mathcal{F})}$
on $\operatorname{Spec} B$

But under any (affine) base change

e.g.
$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} B' \arrow[r] & \operatorname{Spec} B \end{tikzcd}$$
$\mathcal{F}' \quad \mathcal{F}$

$H^p(X',\mathcal{F}') = \text{coho}\left(K^{p-1}\otimes_B B' \to K^p \otimes_B B' \to K^{p+1}\otimes_B B'\right)$

The idea is easy! Choose a finite cover of X by affine open sets, and consider the Čech complex for that cover:

$$0 \to C^0 \to C^1 \to C^2 \to \cdots \to C^n \to 0$$

This is pretty good already — it computes the cohomology of $\mathcal{F}$, indeed under any (affine) base change! Unfortunately, C^i is not finite rank and free.

Just build it from right to left:

$$\begin{tikzcd}
? & ? & ? & ? & ? & B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] & {} \\
0 \arrow[r] & C^0 \arrow[r] & \cdots \arrow[r] & C^{n-2} \arrow[r] & C^{n-1} \arrow[r] & C^n \arrow[r] & 0
\end{tikzcd}$$

inductive step: assume the top is a complex, and we have an isomorphism on cohomology above step m, and a surjection on cohomology at step m:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & B^{\oplus ?} \arrow[r] & B^{\oplus ?} \arrow[r] \arrow[d] & \cdots \\
C^{j-1} \arrow[r] & C^{j} \arrow[r] & C^{j+1} \arrow[r] & C^{j+2} &
\end{tikzcd}
$$

on colio,

So now we have a quasi-isomorphism of complexes of fbt B-modules:

$$\begin{tikzcd}
\arrow[r] & B^{\oplus ?} \arrow[r] \arrow[d] & \cdots & \cdots B^{\oplus ?} \arrow[r] \arrow[d] & B^{\oplus ?} \arrow[r] \arrow[d] & B^{\oplus ?} \arrow[r] \arrow[d] & 0. \\
\arrow[r] & 0 \arrow[r] & \cdots K^0 & \cdots K^{n-2} \arrow[r] & K^{n-1} \arrow[r] & K^n \arrow[r] & 0
\end{tikzcd}$$

Claim: This remains an quasi-isomorphism (iso on cohomology) for any base change / ring map $B \to B'$

Translation: The "single complex" of flat modules

$$\cdots \to K^{n-1} \oplus B^{?} \to K^n \to 0$$

(exactness = quasi-isomorphism) remains exact upon $\otimes_B B'$

Claim: This remains an quasi-isomorphism (iso on cohomology) for any base change / ring map $B \to B'$

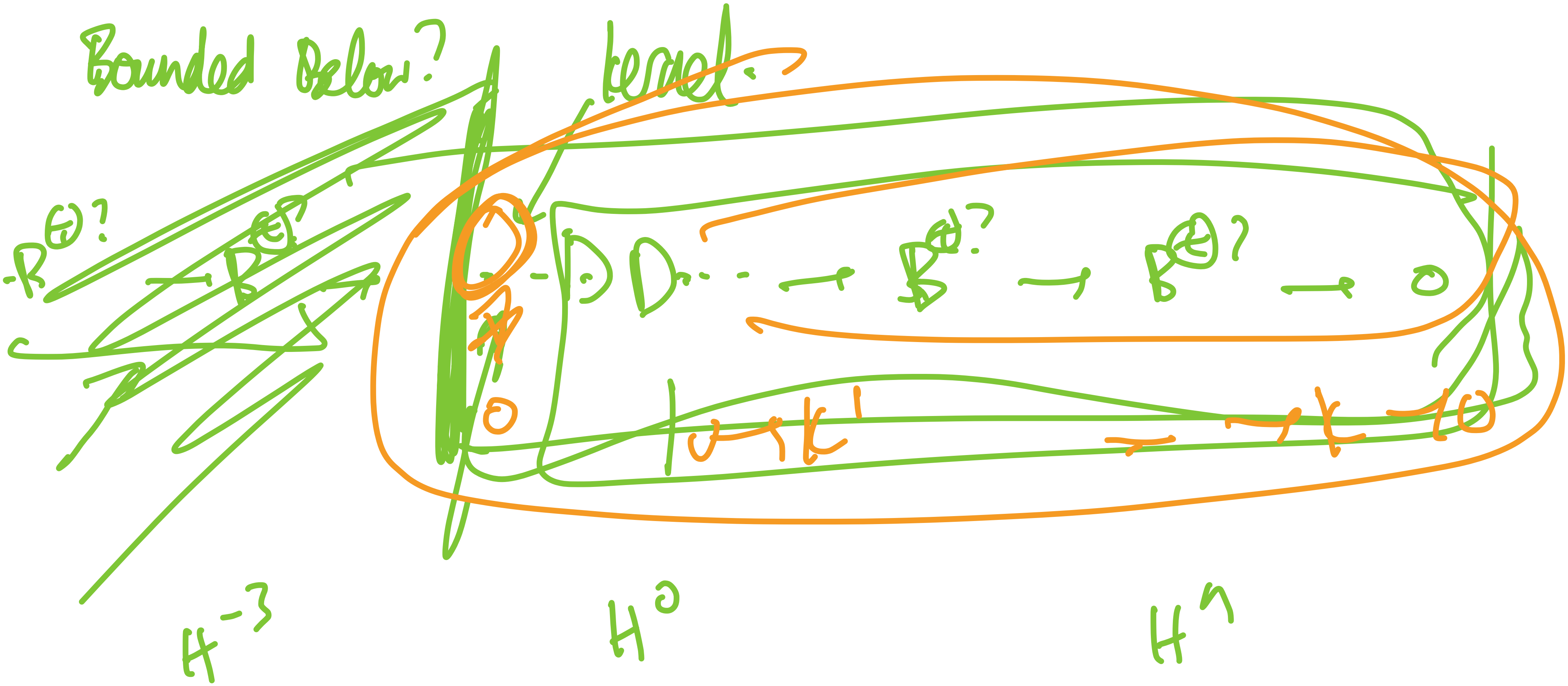
Note: A map of complexes induces an isomorphism on cohomology (i.e., is a quasi-isomorphism) iff the "total complex" is exact. (exercise)

Translation of Claim: The "total complex" of flat modules

$$\cdots \to K^{n-1} \otimes B^{\cdot} \to K^n \to 0$$

remains exact upon $\otimes_B B'$. But exact sequences of flat modules remain exact upon any $\otimes B'$. //

Bounded Below? kernel?



Semicontinuity theorem:

$$B^{\oplus a} \xrightarrow{\alpha} B^{\oplus b} \xrightarrow{\beta} B^{\oplus c} \quad \text{complex.}$$

over Spec B.

dim cohomology is upper semicts function of $p \in \operatorname{Spec} B$.

$$\dim H = \dim \ker \beta - \dim \operatorname{im} \alpha$$

$$= b - \dim \operatorname{im} \beta - \dim \operatorname{im} \alpha$$

$$= b - \operatorname{rank} \beta - \operatorname{rank} \alpha.$$

Now let's prove Cohomology and Base Change following Eric Larson (2020).

We begin with some easy observations about maps of finite rank locally free sheaves $\mathcal{E} \xrightarrow{f} \mathcal{F}$ on a scheme X

We say f is weakly of constant rank a if for every point $p \in X$, $f|_p : \mathcal{E}|_p \to \mathcal{F}|_p$ has rank a.

We say f is **strongly of constant rank a** if for every point $p \in X$, there is an open neighborhood U of p with

$$\begin{tikzcd}
\mathcal{E}|_U \arrow[rr, "f|_U"] \arrow[d, "\sim"] & & \mathcal{F}|_U \arrow[d, "\sim"] \\
\mathcal{O}_U^{\oplus(a+b)} \arrow[rr] \arrow[dr, two heads] & & \mathcal{O}_U^{\oplus(a+c)} \\
& \mathcal{O}_U^{\oplus a} \arrow[ur, hook] &
\end{tikzcd}$$

"what you think"

Note: "strongly of constant rank":

- implies weakly of constant rank
- preserved by base change
- image, kernel, cokernel locally free; and their construction commutes with base change.

Lemma $\operatorname{coker} f$ is locally free iff f is strongly of constant rank

Proof of the "other" direction.

If the cokernel is locally free:

$$\mathcal{E} \xrightarrow{f} \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

Exercise:

$$\mathcal{O}^{\oplus(c+4)} \qquad \mathcal{O}^{\oplus(r+c)} \longrightarrow \mathcal{O}^{\oplus c}$$

$$0 \longrightarrow \mathcal{O}^{\oplus n} \longrightarrow \mathcal{O}^{\oplus(r+c)}$$

Corollary If X is reduced, weakly of constant rank

$=$ strongly of constant rank

Proof:

Lemma Fix $p \in X$. $\ker(f) \to \ker(f|_p)$ is surjective if and only if f is strongly of constant rank in some neighborhood of p.

Proof of Grauert's Theorem

$$\boxed{h^p = \dim K^p - \operatorname{rank} \alpha - \operatorname{rank} \beta}$$

$K^{p-1} \xrightarrow{\alpha} K^p$ and $K^p \xrightarrow{\beta} K^{p+1}$ are weakly of constant rank.

$\therefore$ strongly of constant rank

$\therefore$ Grauert.