Moduli Spaces in Algebraic Geometry

Math 245A (Winter 2022)

Jan. 31, 2022.

Castelnuovo–Mumford regularity

The situation

$$\begin{tikzcd} & \mathcal{F} \arrow[d, dash] & \mathcal{O}(1) \arrow[dl, dash] \\ \mathbb{P}^n & X \arrow[l, hook'] \arrow[d] & \\ & \operatorname{Spec} \bar{k} & \end{tikzcd}$$

Serre Vanishing:

$$H^i(X, \mathcal{F}(m)) = 0 \text{ for } m \gg 0$$
$$\text{for all } i > 0$$

$\mathcal{F}$ is *m-regular* if

$$H^i(X, \mathcal{F}(m-i)) = 0$$
$$\text{for all } i > 0$$

Remark: $\mathbb{P}^n$ $\bar{k}$

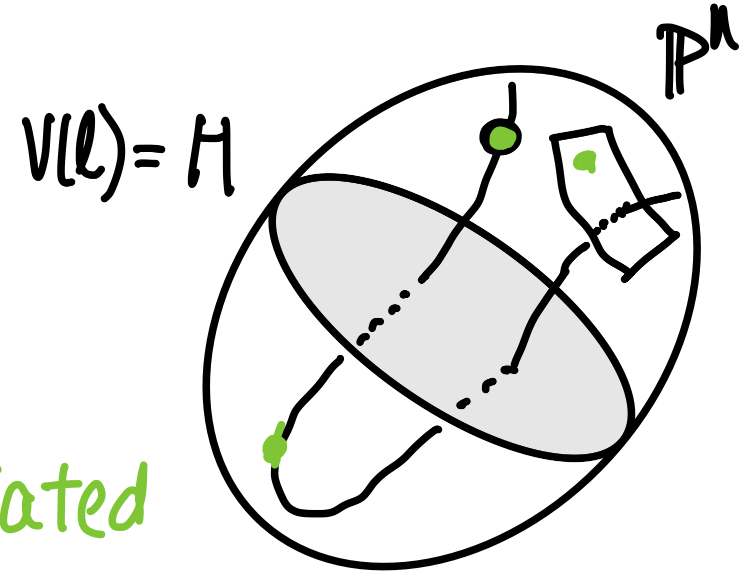
$\mathcal{F}$ is **m-regular** if

$$H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$$

for all $i > 0$

Example: $H \cong \mathbb{P}^{n-1} = V(\ell)$

$$0 \to \mathcal{F}(-1) \xrightarrow{\times \ell} \mathcal{F} \to \mathcal{F}|_H \to 0$$



i.e., H misses the associated points of $\mathcal{F}$

i.e. ℓ is nonzero at the associated points of $\mathcal{F}$.

Then twisting by $m-i$ and taking the long ex. seq.:

$$\underbrace{H^i(\mathbb{P}^n, \mathcal{F}(m-i))}_{0} \to \overset{\therefore\, =0}{H^i(\mathbb{P}^{n-1}, \mathcal{F}|_H(m-i))} \to \underbrace{H^{i+1}(\mathbb{P}^n, \mathcal{F}(m-(i-1)))}_{0}$$

$\therefore\ \mathcal{F}|_H$ is also m-regular.

Note: If k is infinite, you can always find such an H missing the associated points of $\mathcal{F}$.

Indeed, you can always find a hyperplane missing any given finite subset of points. (Why?)

But we took $k = \bar{k}$, which is infinite!

Proposition Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent sheaves on $\mathbb{P}^n$.

If $\mathcal{F}'$ is m-regular
and $\mathcal{F}''$ is m-regular
then $\mathcal{F}$ is m-regular.

Recall

Variations:

If $\mathcal{F}'$ is $(m+1)$-regular
and $\mathcal{F}$ is m-regular
then $\mathcal{F}''$ is m-regular.

If $\mathcal{F}''$ is $(m-1)$-regular
and $\mathcal{F}$ is m-regular
then $\mathcal{F}'$ is m-regular.

$\mathcal{F}$ is m-regular if

$H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$

for all $i > 0$

Proposition If $\mathcal{F}$ on $\mathbb{P}^n$ is m-regular then $\mathcal{F}$ is $(m+1)$-regular.

Proof By induction on n!

Base case $n=0$. Everything is m-regular.

Inductive step. Suppose $n>0$. Goal: $H^i(\mathcal{F}(m+1-i)) = 0$

Choose H so that:

Know $H^i(\mathcal{F}(m-i)) = 0$

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}|_H \to 0$$

$$H^i(\mathcal{F}(m-i)) \to H^i(\mathcal{F}(m-i+1)) \to H^i(H, \mathcal{F}(m-i+1))$$

0

$\therefore = 0.$

$=0$?

$=0$

$\mathcal{F}|_H$ m-regular $\Rightarrow$ $(m+1)$-regular $\Rightarrow$

//

Proposition Suppose $\mathcal{F}$ on $\mathbb{P}^n$ is m-regular. Then

$$H^0(\mathcal{O}(1)) \otimes H^0(\mathcal{F}(r)) \xrightarrow{\mu} H^0(\mathcal{F}(r+1))$$

is surjective, for $r \geq m$.

> $\mathcal{F}$ is m-regular if
> $$H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0$$
> for all $i>0$

Proof by induction on n.

Base case: $n=0$. Reason: $k \otimes V \xrightarrow{\sim} V$.

Inductive step: $n>0$.

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}|_H \to 0$$

$$H^0(H, \mathcal{O}(1)) \otimes H^0(H, \mathcal{F}|_H(r)) \overset{\mu}{\twoheadrightarrow} H^0(\mathcal{F}|_H(r+1))$$

$r \geq m$ $\mathcal{F}$ m-regular

$0 \to \mathcal{F}(-1) \xrightarrow{\times \ell} \mathcal{F} \to \mathcal{F}|_H \to 0$

$H^0(\mathbb{P}^n, \mathcal{F}(r))$

$\downarrow \times \ell$ linear form

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{F}(r)) & \xrightarrow{\mu_n} & H^0(\mathbb{P}^n, \mathcal{F}(r+1)) \\ \downarrow & & \downarrow \\ H^0(\mathbb{P}^{n-1}, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^{n-1}, \mathcal{F}|_H(r)) & \xrightarrow{\mu_{n-1}} & H^0(\mathbb{P}^{n-1}, \mathcal{F}|_H(r+1)) \end{array}$$

surjective??

(hypothesis)

$\downarrow$

$H^1(\mathbb{P}^n, \mathcal{F}(r)) = 0$

Three of four maps are surjective. Write one kernel.

Clever trick: $H^0(\mathbb{P}^n, \mathcal{F}(r+1)) = \operatorname{im}(\mu_n) + \operatorname{im} H^0(\mathbb{P}^n, \mathcal{F}(r))$

$= \operatorname{im}(\mu_n)$

$\operatorname{im} H^0(\mathbb{P}^n, \mathcal{F}(r)) \subset \operatorname{im}(\mu_n)$

Proposition Suppose $\mathcal{F}$ on $\mathbb{P}^n$ is m-regular.
Then $\mathcal{F}(m)$ is generated by global sections.

Proof By induction on n again.

Base case $n=0$: ✓ points are affine.

Inductive step:

Wish to show: $H^0(\mathcal{F}(r)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{F}(r)$

is a surjective map of coherent sheaves.

How can you check if a map $\mathcal{G} \xrightarrow{\alpha} \mathcal{H}$ of coherent sheaves on $\mathbb{P}^n$ is surjective?

Exercise: if and only if $H^0(\mathcal{G}(s)) \xrightarrow{\alpha} H^0(\mathcal{H}(s))$ is surjective for $s \gg 0$.

(Hint: Show that if $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$ is an exact sequence of sheaves on $\mathbb{P}^n$ then for $s \gg 0$, $0 \to H^0(\mathcal{G}_1(s)) \to H^0(\mathcal{G}_2(s)) \to H^0(\mathcal{G}_3(s)) \to 0$ is exact.

Then show that if $0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_N \to 0$ is an exact sequence of sheaves on $\mathbb{P}^n$ then for $s \gg 0$, $0 \to H^0(\mathcal{G}_1(s)) \to \cdots \to H^0(\mathcal{G}_N(s)) \to 0$ is exact. Apply this to $0 \to \ker \alpha \to \mathcal{G} \to \mathcal{H} \to \operatorname{coker} \alpha \to 0$.)

Recall:

Proposition Suppose $\mathcal{F}$ on $\mathbb{P}^n$ is m-regular.
Then $\mathcal{F}(m)$ is generated by global sections.

Proof By induction on n again.

~~Base case $n=0$:~~ (You did it.) for no reason.

~~Inductive step:~~

Wish to show: $H^0(\mathcal{F}(r)) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F}(r)$

is a surjective map of coherent sheaves.

So for $s \gg 0$, we wish to show:

$$H^0\big(H^0(\mathcal{F}(r)) \otimes \mathcal{O}_{\mathbb{P}^n}(s)\big) \to H^0(\mathcal{F}(r+s)) \text{ is surjective}$$

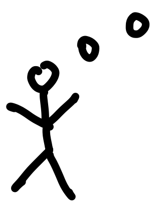
i.e. $$H^0(\mathcal{F}(r)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(s)) \to H^0(\mathcal{F}(r+s)) \text{ is surjective.}$$

Okay by induction on s! //

<u>Upshot</u>: We have proved, among other useful things, that if $\mathcal{F}$ is m-regular, then $\mathcal{F}(r)$ is generated by global sections, and has no higher cohomology, for all $r \geq m$. Woo hoo!

Big Theorem of Today

Given: $n \in \mathbb{Z}^{\geq 0}$, $\rho \in \mathbb{Z}^{>0}$, $p(t) \in \mathbb{Q}[t]$



"$\mathbb{P}^n$" "rank" Hilbert polynomial

Then there is some $m = m(n, \rho, p(t))$ such that for *any* coherent sheaf $\mathcal{F} \hookrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus \rho}$ with Hilbert polynomial $p(t)$, $\mathcal{F}$ is m-regular.

Proof by induction on n.

Base case $n = 0$: Everything is regular!

Inductive step: $n > 0$.

Replace k by $\bar{k}$ as usual.

Define $\mathcal{G}$ by $0 \to \boxed{\mathcal{F} \to \mathcal{O}^{\oplus \ell}} \to \mathcal{G} \to 0$

$\otimes\, \mathcal{O}_H$

$0 \to \mathcal{F} \to \mathcal{O}^{\ell} \to \mathcal{G} \to 0$

$0 \to \mathcal{F}|_H \to \mathcal{O}_H^{\ell} \to \mathcal{G}|_H \to 0$

Choose hyperplane H missing the associated points of $\mathcal{G}$ and $\mathcal{F}$.

$0 \to \mathcal{F}(-1) \xrightarrow{\times \ell} \mathcal{F} \to \mathcal{F}|_H \to 0$

From $0 \to \mathcal{F}(t-1) \xrightarrow{\times \ell} \mathcal{F}(t) \to \mathcal{F}|_H(t) \to 0$

the Hilbert polynomial of $\mathcal{F}|_H$ is known:

it is: $\chi(\mathcal{F}|_H(t)) = p(t) - p(t-1)$

$\mathcal{F}|_H$ is a coh. sheaf on $\mathbb{P}^{n-1} \cong H$.

Thus $\mathcal{F}|_H$ is m'-regular for some m' by ind. hyp. (depends on $P_{\mathcal{F}}, n_{\mathcal{F}}, \rho_{\mathcal{F}}$)

Then for $m \geq m'-1$:

$$0 \to \mathcal{F}(m-1) \to \mathcal{F}(m) \to \mathcal{F}|_H(m) \to 0$$

The long exact sequence in cohomology yields

$$\underset{0}{\cancel{H^{\overset{\geq 1}{s-1}}(\mathcal{F}|_H(m))}} \to H^s(\mathcal{F}(m-1)) \xrightarrow{\cong} H^s(\mathcal{F}(m)) \to \underset{0}{\cancel{H^{\overset{\geq 2}{s}}(\mathcal{F}|_H(m))}}$$

$\boxed{s \geq 2:}$

Serre vanishing $m \gg 0$ $H^s(\mathcal{F}(m)) = 0$.

$$\underset{0}{\underset{\parallel}{H^s(\mathcal{F}(m-1))}} = H^s(\mathcal{F}(m)) = H^s(\mathcal{F}(m+1)) = \cdots = 0$$

Almost there! $H^2(\mathcal{F}(m-1)) = H^3(\mathcal{F}(m-1)) = \cdots = 0$.

What about $\underline{s=1}$? Is $\underline{H^1(\mathcal{F}(m-1)) = 0}$?

$$0 \to \mathcal{F}(m-1) \longrightarrow \mathcal{F}(m) \longrightarrow \mathcal{F}|_H(m) \longrightarrow 0$$

surjective ??   may not be zero!

$$H^0(\mathcal{F}(m)) \longrightarrow H^0(\mathcal{F}|_H(m))$$

$$\to H^1(\mathcal{F}(m-1)) \twoheadrightarrow H^1(\mathcal{F}(m)) \longrightarrow \cancel{H^1(\mathcal{F}|_H(m))} \to 0$$

$$\begin{tikzcd}
H^0(\mathcal{F}(m-1)) \arrow[r, "\alpha_{m-1}"] \arrow[d] & H^0(\mathcal{F}(m-1)|_H) \arrow[d] \\
H^0(\mathcal{F}(m)) \arrow[r, "\alpha_m"] & H^0(\mathcal{F}(m)|_H)
\end{tikzcd}$$

$H^1(\mathcal{F}(m)) \twoheadrightarrow H^1(\mathcal{F}(m+1))$

$H^1(\mathcal{F}(m)) = 0$

for $m \gg 0$.

α_{m-1} surjective $\Longrightarrow$ $\alpha_m, \alpha_{m+1}, \ldots$ surjective.

$\therefore$ $h^1(\mathcal{F}(m'))$ might not be zero, but if we could bound it by B, we know $H^1(\mathcal{F}(m'+B)) = 0$

So last but not least, we want a bound $h^1(F(m'))$ depending only on $p(t)$, n, r.

Now $p(m') = h^0(F(m')) - h^1(F(m')) + \underbrace{h^2(F(m')) - \cdots}_{=0}$

$$h^1(F(m')) = -p(m') + h^0(F(m')) \qquad F \subseteq \mathcal{O}^{\oplus \rho}$$

$$\leq -p(m') + h^0(\mathbb{P}^n, \mathcal{O}^{\oplus \rho}(m'))$$

$$= -p(m') + \rho\binom{m'+n}{n}$$

QED!