

Moduli Spaces in Algebraic Geometry

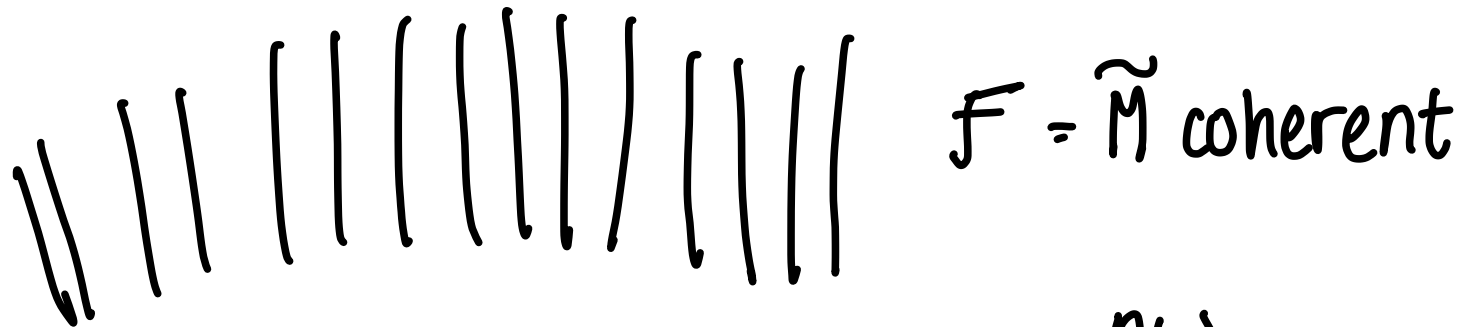
Math 245 A (winter 2022)

Jan. 28, 2022.

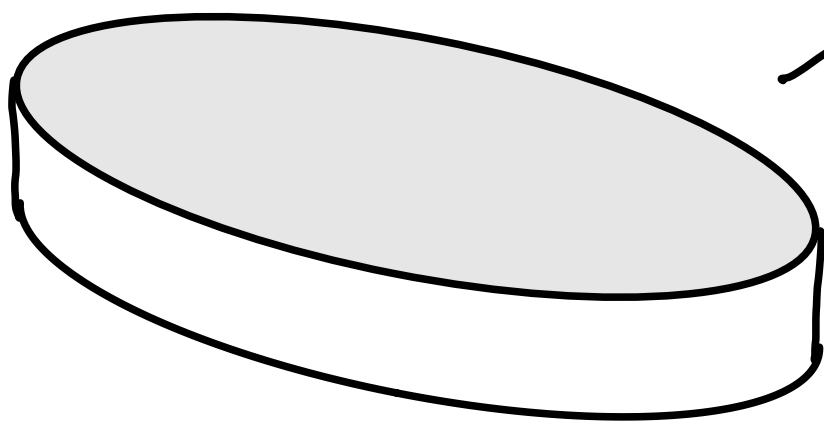


the
flattening
stratification

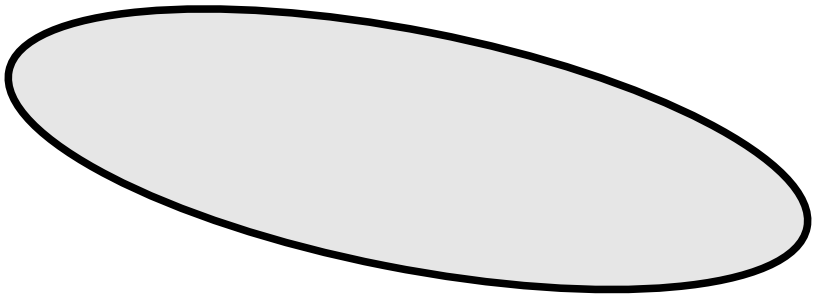
The Situation



$\mathcal{O}(1)$



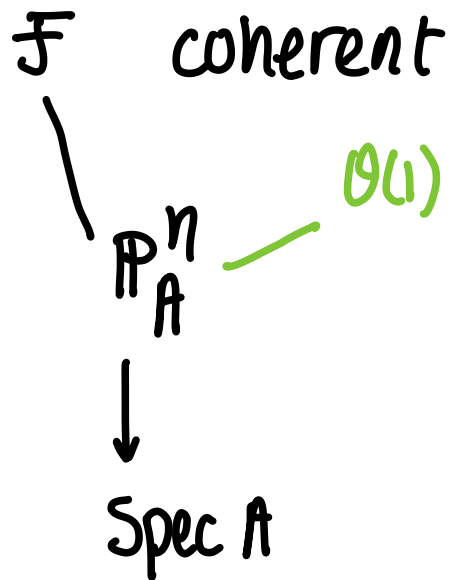
\mathbb{P}^n_A



Spec A
Noetherian

Our goal:
Stratify
into "best"
locally closed
subschemes
over which F
is flat

The players:



The tools:

- twist by $\mathcal{O}(1)$
- (higher) pushforward (from \mathbb{P}_A^n)
- pullback (from $\text{Spec } A$)
(to points? affines?
locally closed? arbitrary?)
- flattening stratification of coherent sheaf on $\text{Spec } A$
- Serre Vanishing

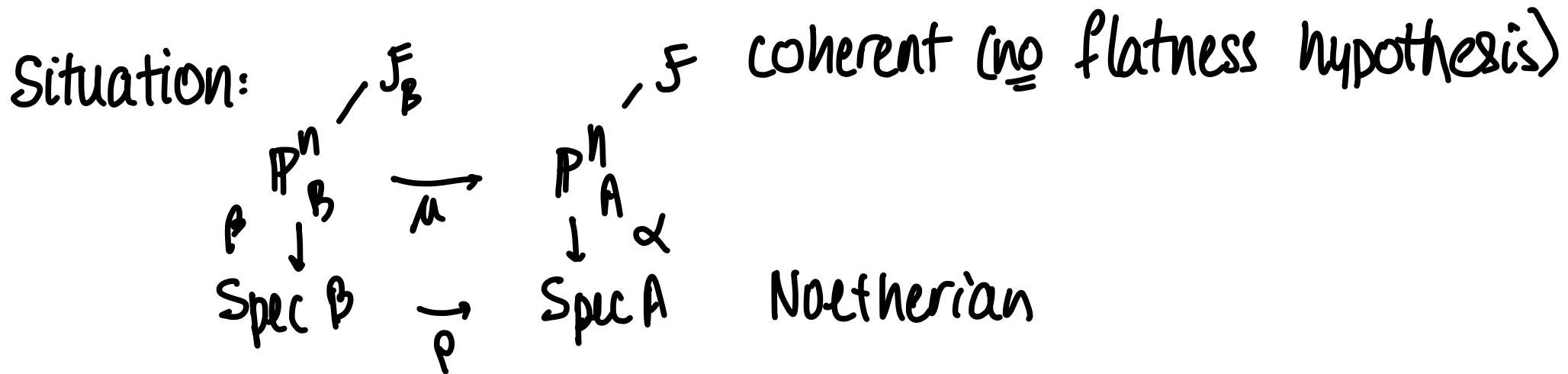
Questions:

Is higher pushforward/cohomology zero?

Is a quasicohherent sheaf upstairs flat/base?

Is a quasicohherent sheaf downstairs flat/base
= vector bundle?

Reminder: First Lemma



Then there

is some M so that for $m \geq M$, $\rho^* R^i \alpha_* \mathcal{F}(m) \rightarrow R^i \beta_* \mu^* \mathcal{F}(m)$ is an isomorphism.

Real content: $i=0$

$$\rho^* \alpha_* \mathcal{F}(m) \rightarrow \beta_* \mu^* \mathcal{F}(m)$$

PROOF OF THE FLATTENING STRATIFICATION:

First few steps get us to:

If you twist up enough,
cohomology commutes with base change to every point.

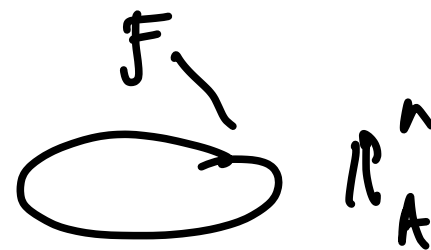
Precisely: there exists M such that for every $m \geq M$,
and every $p \in \text{Spec } A$,

$$k(p) \otimes_A H^i(\mathbb{P}_A^n, \mathcal{F}(m)) \longrightarrow H^i(\mathbb{P}_p^n, \mathcal{F}|_{\mathbb{P}_p^n}(m))$$

is an isomorphism.

Reminder of how we got this.

"generic flatness"

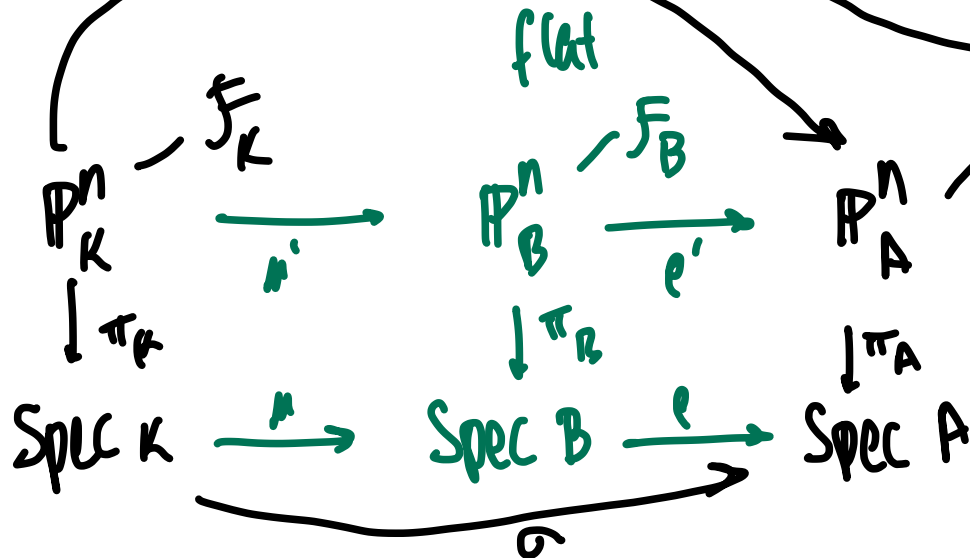


point

partition into
locally closed subsets



σ'



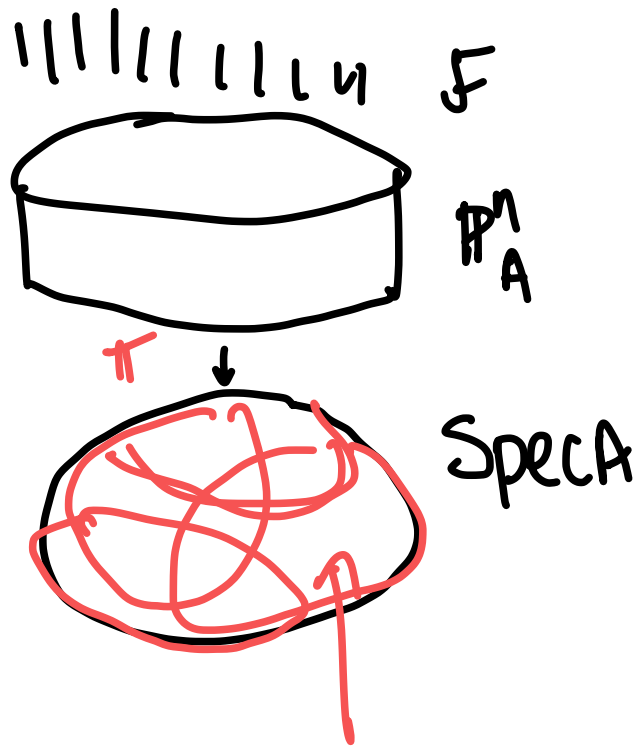
$$(\sigma')^* R^i(\pi_A)_* F(m) \rightarrow$$

$$R^i(\pi_A)_* (\sigma')^* F(m)$$

Then, we found the (candidate) topological stratification.

Idea: indexed by Hilbert polynomial of the fibers.

$$\in \mathbb{Q}[t] \quad \text{degree} \leq n$$



What to do?

$$\begin{array}{c} \checkmark \\ \pi_* \mathcal{F}(M) \\ \vdots \\ \pi_* \mathcal{F}(M \otimes \mathcal{O}(n)) \end{array}$$

rest. to any pt.

semicontinuity
of rank of coherent sheaf.

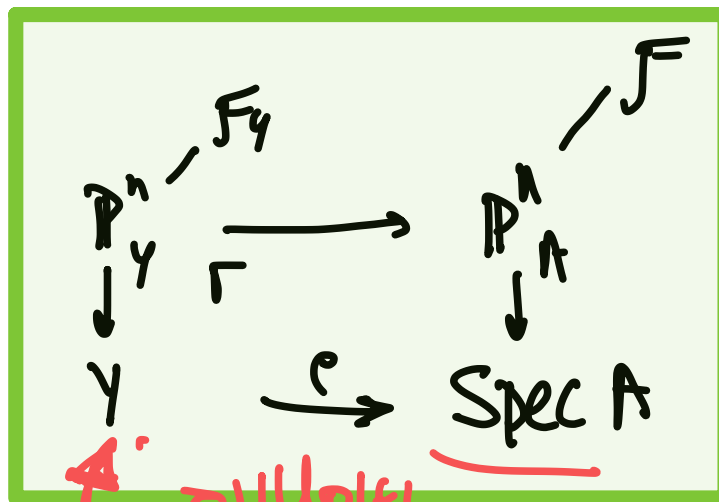
Remark (in anticipation of universal property):

Suppose we have

$$\rho: Y \rightarrow \text{Spec } A$$

such that

\mathcal{F}_Y is flat over Y .



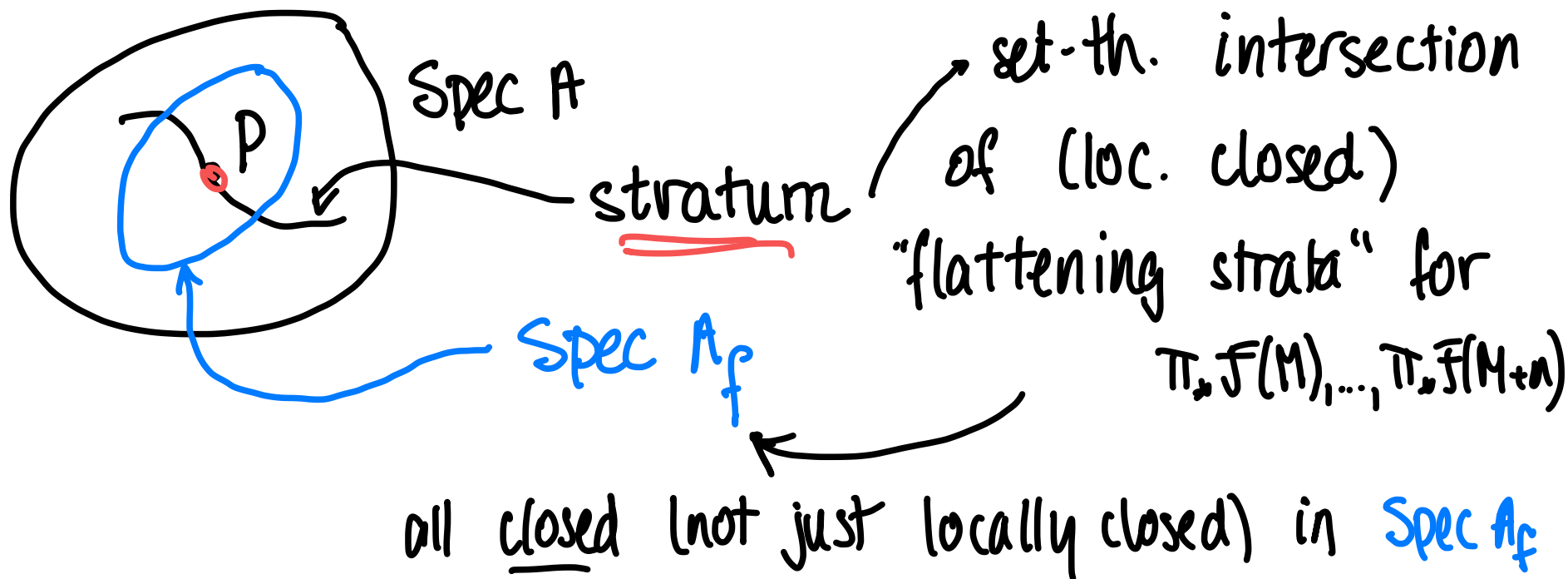
Then set-theoretically, $|Y| \xrightarrow{\rho} |\text{Spec } A|$ factors through our "stratification by Hilbert polynomial." Why?

Fourth step: get the (candidate) scheme structure on these strata.

Consider $\pi_* \mathcal{F}(M), \dots, \pi_* \mathcal{F}(M+n)$ on $\text{Spec } A$, which we used to find the strata.

initely many strata!

Near a point P on one of these strata:



In $\text{Spec } A_f$:
ideals:

$\text{Spec } A_f / I_0$

locally free.
↙

$I_0 \sim$ flattening closed subscheme for $\pi_* \mathcal{F}(M)$

$I_1 \sim$ flattening closed subscheme for $\pi_* \mathcal{F}(M+1)$ ✓

so $I_0 + I_1 \sim$ biggest closed subscheme of
 $\text{Spec } A_f$ where $\pi_* \mathcal{F}(M)$ and
 $\pi_* \mathcal{F}(M+1)$ are locally free of desired rank.

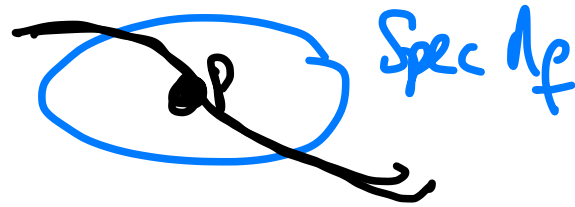
$I_n \sim$ flattening closed subscheme for $\pi_* \mathcal{F}(M+n)$ ✓

$I_0 + \dots + I_n \sim$ biggest closed subscheme of

$\text{Spec } A_f$ where $\pi_* \mathcal{F}(M), \dots$

$\pi_* \mathcal{F}(M+n)$ are locally free of desired rank.

Keep going!



I_{n+1}, I_{n+2}, \dots flattening subscheme for $\pi_* \mathcal{F}(M+n+1),$
 $\pi_* \mathcal{F}(M+n+2), \dots$

(minor caution here — what if
these are locally closed but not closed?)

so consider

$$I_0 + \dots + I_{n+1} \subseteq I_0 + \dots + I_{n+2} \subseteq \dots = I_0 + \dots + I_{\infty}$$

which eventually stabilizes (Noetherianity!), to I , say.

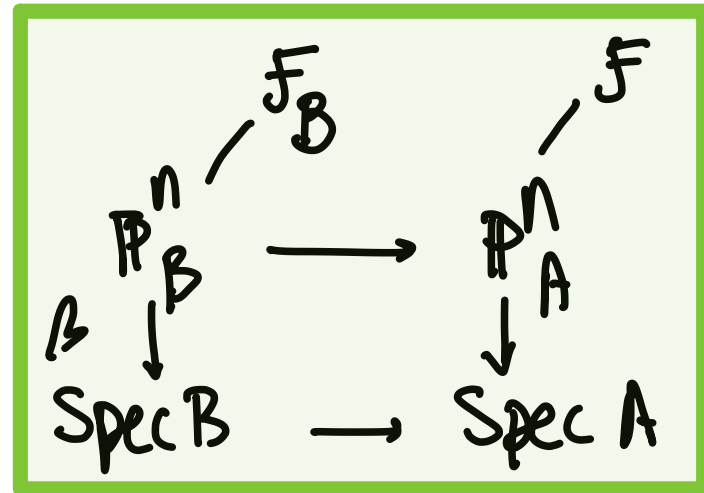
Fifth Step

Spec A_f

Spec $B = \text{Spec } A_f / I$

F is flat over this $\text{Spec } A_f / I$.

$$I = I_0 + I_1 + \dots$$



Reason: For this fixed $\text{Spec } B$,

(higher) pushforwards of $\mathbb{F}(m)$

"First Lemma"

commute with base change for $m \geq M'$ for some M' .

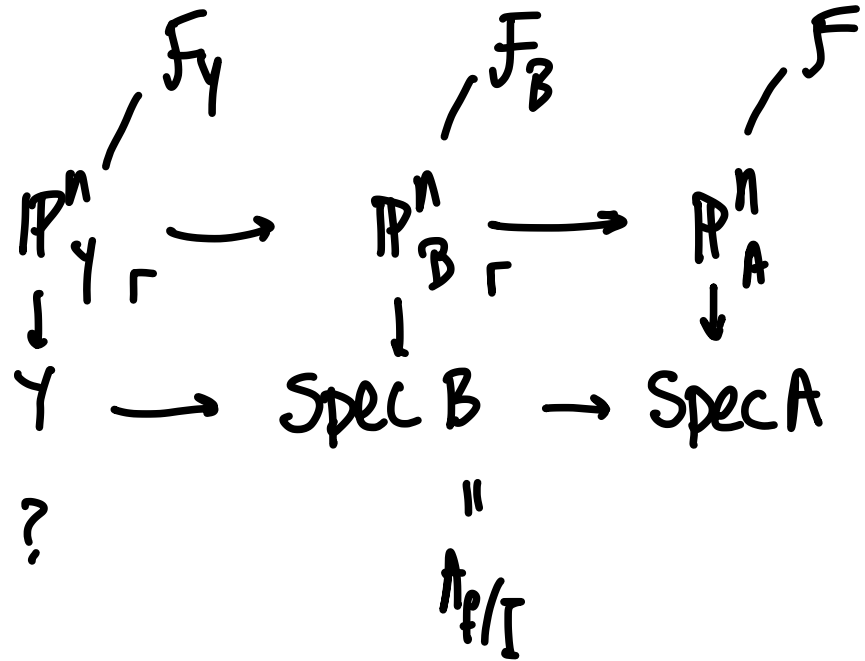
Thus $\beta_* \mathbb{F}_B(m)$ is locally free for $m \gg 0$,

so (by our criterion) \mathbb{F}_B is flat!

Sixth step

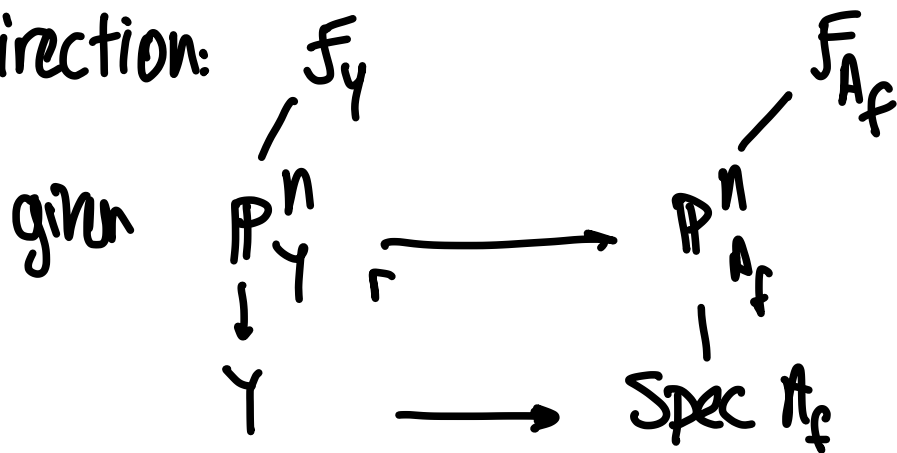
This (on our neighborhood $\text{Spec } A_f$ of p) satisfies our universal property!

Proof: One direction



Why is f_Y flat over Y ?

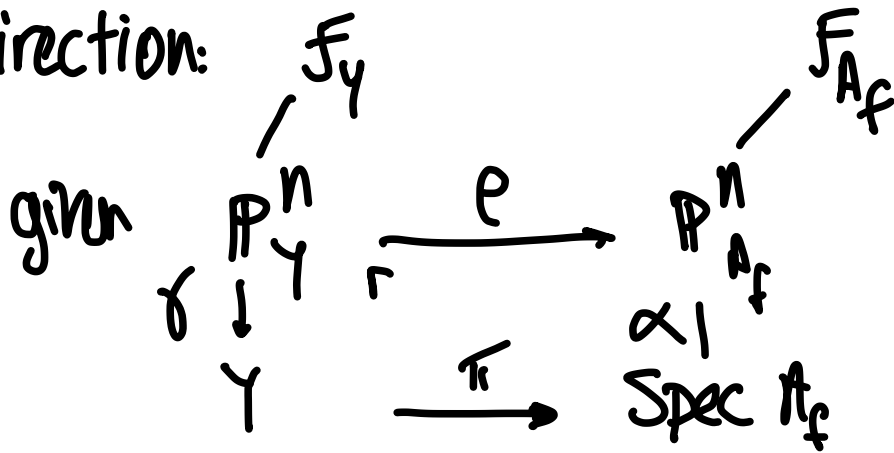
Other direction:



with f_Y flat over Y and the Hilbert pols of fibers = $p(t)$, why does this factor $Y \rightarrow \text{Spec } B \rightarrow \text{Spec } A_f$?

$$\text{Spec } B = \bigcap_{j=0}^{\infty} V(I_j)$$

Other direction:



with \mathcal{F}_Y flat/ Y
 and the Hilbert
 polys of fibers = $p(t)$,
 why does this factor
 $Y \rightarrow \text{Spec } B \rightarrow \text{Spec } A_f$?

Answer: Set-theoretically: ok!

If $\pi^* \alpha_* \mathcal{F}_Y(M+i)$ is locally free of the expected dimension

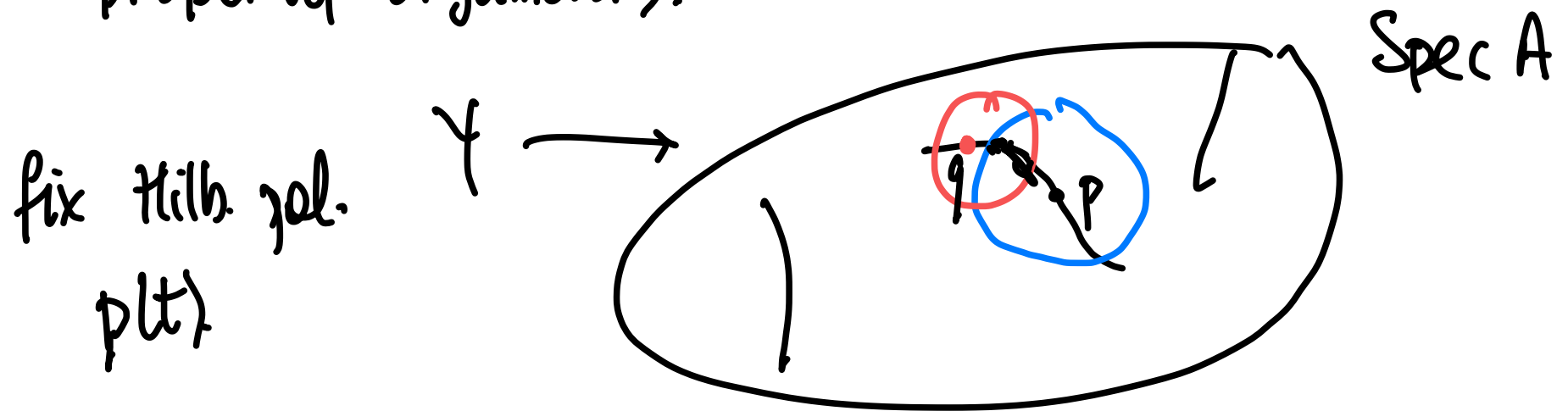
for $i=0,1,\dots$ then by the universal property of the flattening stratification of $\alpha_* \mathcal{F}_Y(M+i)$, $Y \xrightarrow{\pi} \text{Spec } A_f$ factors through $\text{Spec } A_f / I_i$. (Think this through...)

First lemma: $q \Rightarrow 0$

$$\pi^* \alpha_* \mathcal{F}_Y(q) \cong \gamma_* \left(\frac{\rho^* \mathcal{F}_q}{\text{flat}} \right)$$

Seventh step

(that you may not have noticed): all of these constructions glue together (by a universal property argument).



This completes the proof! (Do you agree?)



Castelnuovo -
Mumford
regularity

CASTELNUOVO-MUMFORD REGULARITY

Motivation: ~~\mathbb{A}^n~~ \mathbb{P}^n (over \mathbb{C} , say).

I tell you the Hilbert polynomial of \mathcal{F} , say $p_{\mathcal{F}}(t)$. Then can you tell me some N so that:

for example,

$$H^i(X, \mathcal{F}(N)) = 0 \quad \text{for } i > 0 ?$$

~~(Then $h^0(X, \mathcal{O}(N)) = p(N)$).~~

"effective Serre vanishing based only on the Hilbert polynomial".

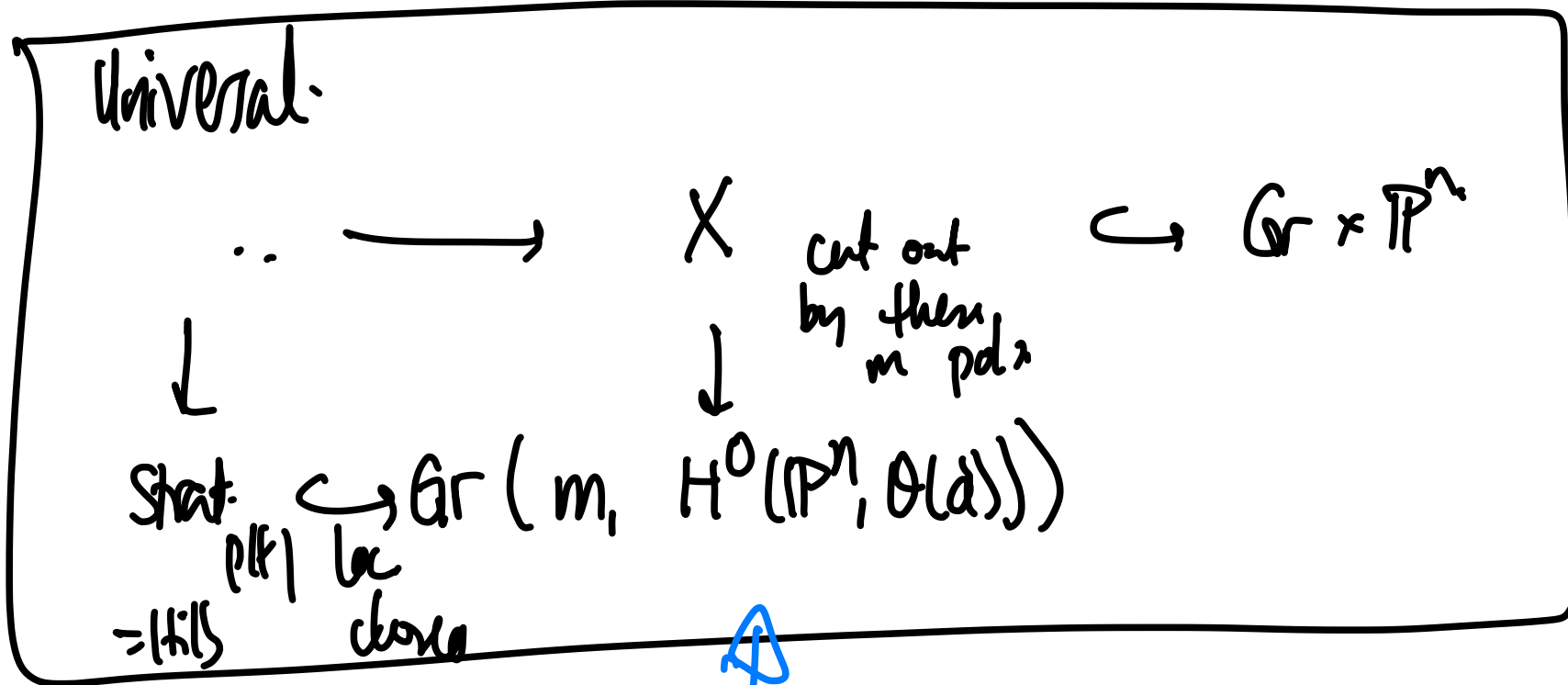
Can you tell me some d, m , depending only on $p(t)$,
so that X lies on precisely an m -dimensional
family of degree d hypersurfaces, i.e.

$$\dim_{\mathbb{C}} \left(\ker \left(H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow H^0(\mathcal{O}_X(d)) \right) \right) = m$$

and these m equations precisely cut
out X ?

Some initial thoughts:

We might hope the Hilbert scheme $\text{Hilb}_{d(t)} \mathbb{P}^n$ might naturally be found using:



Flattening
Strat.



Find d, m

Goal:

$$\ker(H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow H^0(\mathcal{O}_X(d)))$$

Also:

$$0 \rightarrow I_X(d) \rightarrow \mathcal{O}_{\mathbb{P}^n}(d) \rightarrow \mathcal{O}_X(d) \rightarrow 0$$

$$m = h^0(I_X(d)) = \chi(I_X(d))$$

Guess: $H^{i>0}(\mathbb{P}^n, I_X(d)) = 0$

$$\begin{array}{c} H^0(I_X(d)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \\ \rightarrow H^0(\mathcal{O}_X(d)) \rightarrow 0 \end{array}$$

Want

$$I_X \subset \mathbb{C}[x_0, \dots, x_n]$$

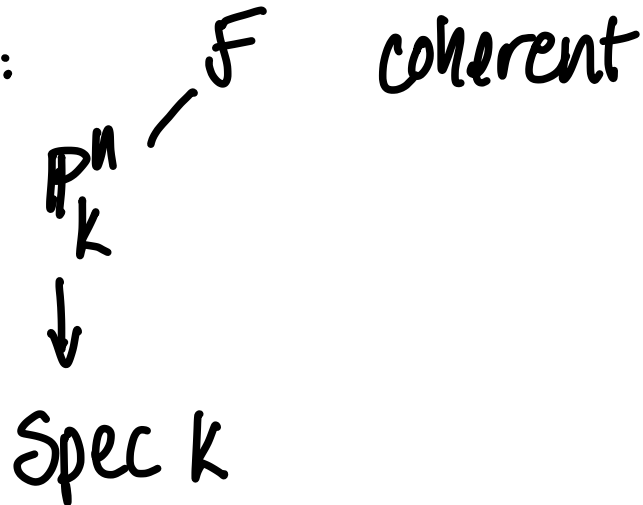
$$(I_X)_d \times \mathbb{C}^{N+1} \rightarrow (I_X)_{d+1}$$

φ

surjective

Let us begin.

Situation:



We say \mathcal{F} is m -regular if

$$H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0 \text{ for } i > 0$$

OR $H^i(\mathbb{P}^n, \mathcal{F}(m'-i)) = 0$ for $i > 0$, $m' \geq m$.

Proposition

(m -regularity and field extension)

$$\begin{array}{ccc} \mathcal{F}_\ell & & \mathcal{F}_k \\ | & & | \\ \mathbb{P}_\ell & \longrightarrow & \mathbb{P}_k \\ \downarrow & & \downarrow \\ \text{Spec } \ell & \longrightarrow & \text{Spec } k \end{array}$$

\mathcal{F} is m -regular iff \mathcal{F}_ℓ is m -regular.

PF

$$H^i(\mathbb{P}_\ell^n, \mathcal{F}_\ell(m-i)) = 0$$

iff

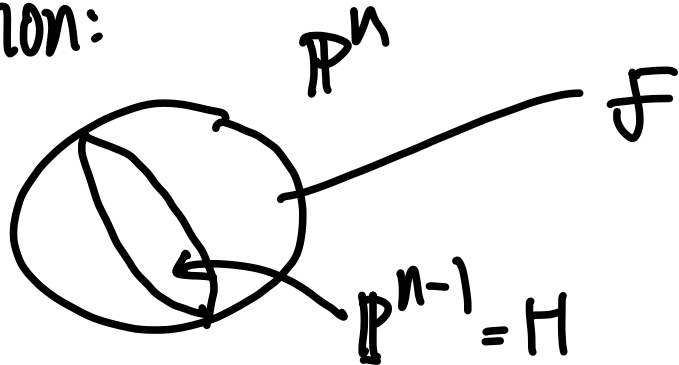
$$H^i(\mathbb{P}_k^n, \mathcal{F}_k(m-i)) = 0$$

for $i > 0$

Bea's Lemma: $\mathcal{F}(-1)$ is $(m+1)$ -regular iff \mathcal{F} is m -regular

Proposition (m -regularity and hyperplane slicing)

Situation:



\mathcal{F} has no associated points
on H

Translation:

$$0 \rightarrow \mathcal{F}(-1) \rightarrow \mathcal{F} \rightarrow \mathcal{F}|_H \rightarrow 0$$

Then \mathcal{F} m -regular $\Rightarrow \mathcal{F}|_H$ m -regular.

Pf. (hold off for a second)

Proposition Suppose $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of coherent sheaves on \mathbb{P}^n .

If \mathcal{F}' is m -regular
and \mathcal{F}'' is m -regular
then \mathcal{F} is m -regular.

Variations:

If \mathcal{F}' is $(m+1)$ -regular
and \mathcal{F} is m -regular
then \mathcal{F}'' is m -regular.

If \mathcal{F}'' is $(m-1)$ -regular
and \mathcal{F} is m -regular
then \mathcal{F}' is m -regular.

Proposition Suppose \mathcal{F} on \mathbb{P}^n is m -regular.

Then:

(a) $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{F}(r)) \rightarrow H^0(\mathbb{P}^n, \mathcal{F}(r+1))$
is surjective.

(b) \mathcal{F} is $(m+1)$ -regular.

(c) $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ for $i > 0$, and

(d) $\mathcal{F}(r)$ is generated by global sections.

Proof: soon, and easy!

$r \geq m$??

Theorem (Mumford)

$$\boxed{0 \rightarrow I \rightarrow \mathcal{O} \rightarrow \mathcal{O} \rightarrow 0}$$

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}^{\oplus r} \rightarrow \dots \rightarrow 0$$

If \mathcal{F} is a coherent sheaf that is a subsheaf of $\mathcal{O}^{\oplus 100}$ on \mathbb{P}^{263} , with known Hilbert polynomial, Mumford can bound its regularity.

Next week: we'll be able to do it too!

$$\begin{array}{ccccccc}
 & & & & & & \downarrow \\
 & & & & & & \mathcal{O}^{\oplus r} \rightarrow \mathcal{G} \rightarrow 0 \\
 & & & & & & \uparrow \\
 & & & & & & \mathcal{O}^{\oplus r}(i) \rightarrow \mathcal{G}(i) \rightarrow 0 \\
 & & & & & & \uparrow \\
 & & & & & & \mathcal{F}(i) \rightarrow \mathcal{O}^{\oplus r}(i) \rightarrow \mathcal{G}(i) \rightarrow 0 \\
 & & & & & & \uparrow \\
 & & & & & & \mathcal{F} \rightarrow \mathcal{O}^{\oplus r} \rightarrow \mathcal{G} \rightarrow 0
 \end{array}$$

$H^j(\mathcal{G}(i)) \simeq H^{j+1}(\mathcal{F}(i)).$

$$0 \rightarrow G \xrightarrow{\text{lip}} \oplus \mathcal{O}(a) \rightarrow F \rightarrow 0$$

$$0 \rightarrow G(a) \rightarrow \oplus \mathcal{O} \rightarrow F(a) \rightarrow 0$$

