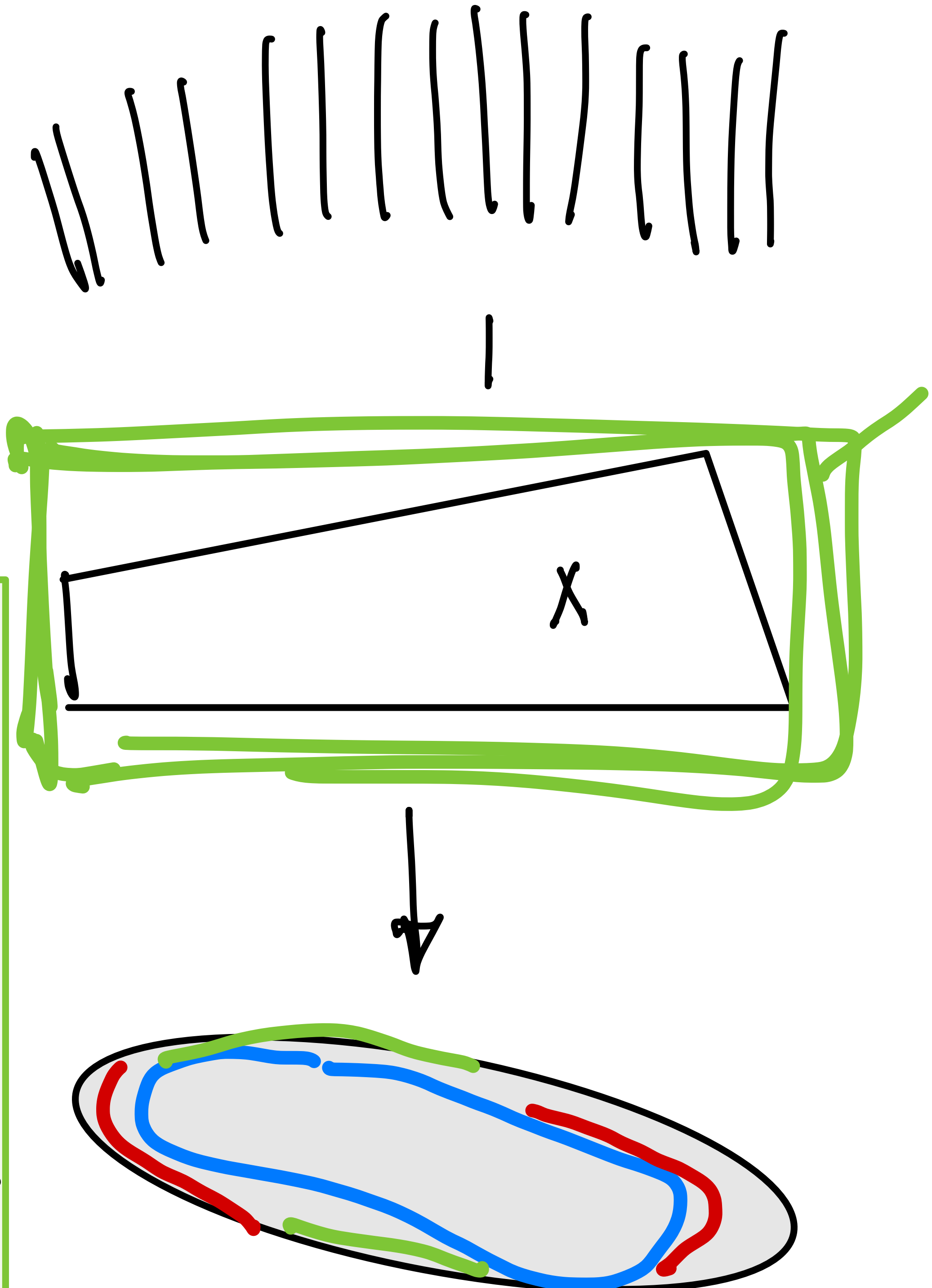
Moduli Spaces in Algebraic Geometry

Math 245A (Winter 2022)

Jan. 26, 2022.

the flattening stratification

The Situation



$\mathcal{F}$ coherent

$\mathcal{O}(1)$

projective

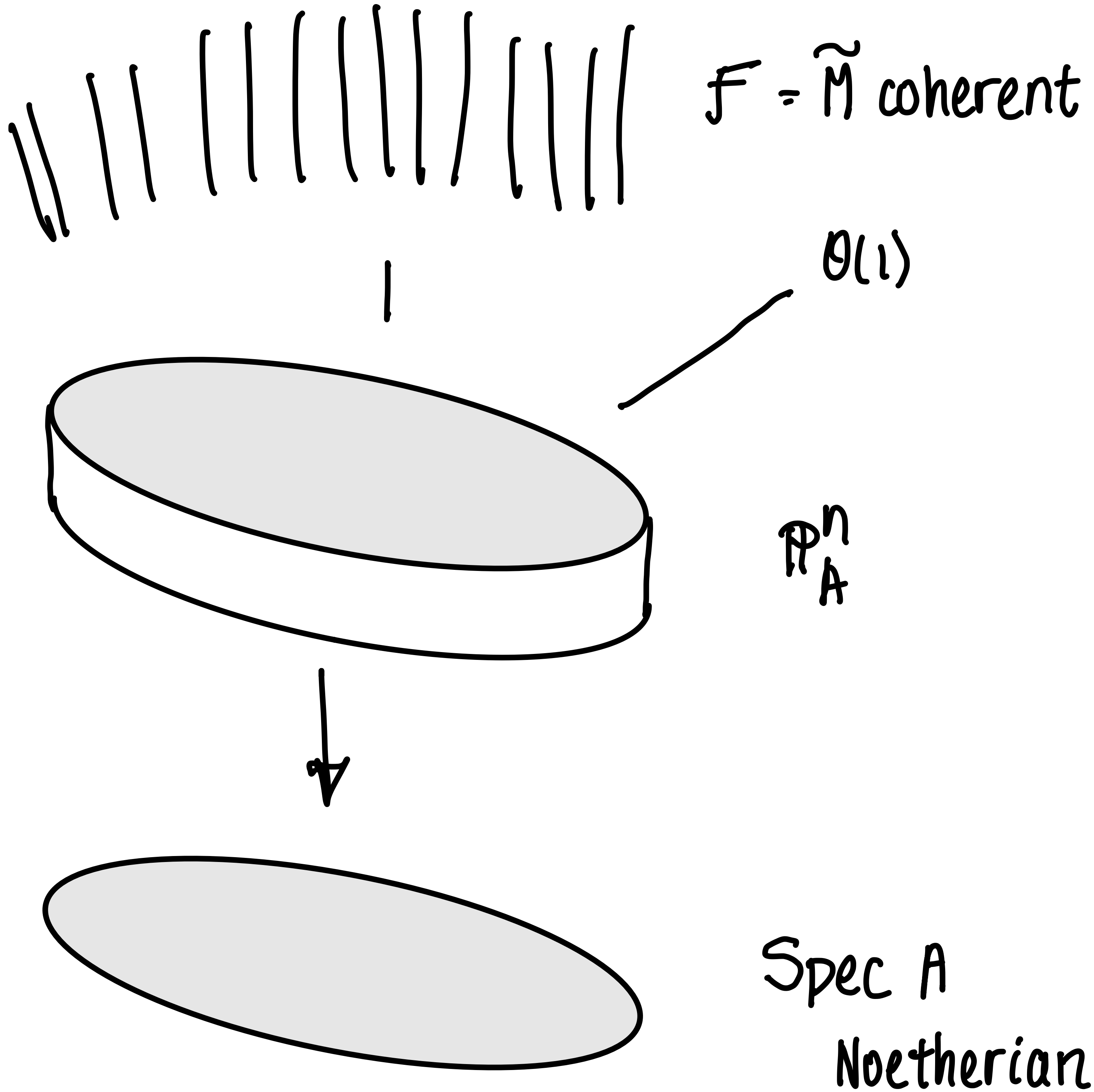
$\mathbb{P}^n_A$.

$S = \operatorname{Spec} A$

locally Noetherian

Our goal: Stratify into "best" locally closed subschemes over which $\mathcal{F}$ is flat

The Situation



Our goal:

Stratify into "best" locally closed subschemes over which $\mathcal{F}$ is flat

Previously...

- We did the case where $\eta = 0$.

Previously...

motivation

- If $\mathcal{F}$ is flat,

for $m \gg 0$, $(R^i\pi_*)\mathcal{F}(m) = 0$ (regardless of flatness — this is Serre vanishing)

$\pi_*\mathcal{F}(m)$ is flat, finite rank locally free, vector bundle

Reason: Čech complex for $\mathcal{F}(m)$:

$$0 \to \underbrace{\overset{H^0}{(\pi_*\mathcal{F}(m))} \to \check{C}^0 \to \underbrace{\check{C}^1 \to \cdots \to \check{C}^n}_{\text{exact}}} \to 0$$

And These are all preserved by any base change.

Reason:

Previously...

Converse: if for all $m \gg 0$, $\exists M > 0$ so that for all $m \geq M$ . . .

$R^i \pi_* \mathcal{F}(m) = 0$ for $i > 0$

(free, from Serre vanishing)

and $\pi_* \mathcal{F}(m)$ is flat ($=$ loc. free finite rank vector bundle etc.)

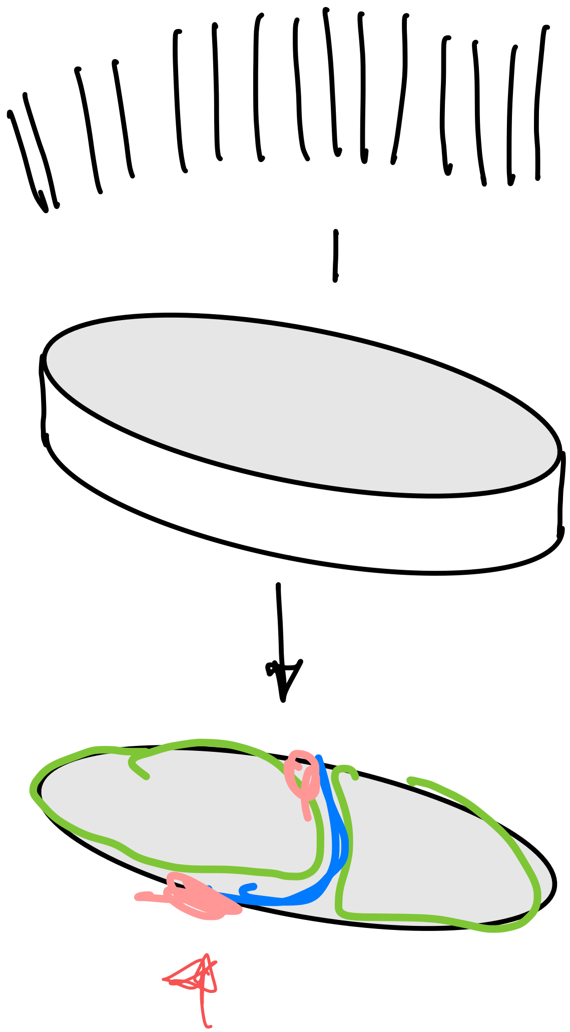
then $\mathcal{F}$ is flat.

Idea of proof:

$M := \bigoplus_{m \gg 0} H^0(\pi_* \mathcal{F}(m))$ is flat A-module, graded $A[x_0, \dots, x_n]$ module Then...

Based on this, here is a plausible strategy that doesn't quite work.

(My question for you: what are the issues?)



$\mathcal{F}(m)$

$\mathbb{P}^n_A$

Spec A Noetherian

Choose M so that for $m \geq M$, $\underbrace{R^i\pi_*\mathcal{F}(m) = 0}$.

$\pi_*\mathcal{F}(m)$ is a coherent sheaf on Spec A.

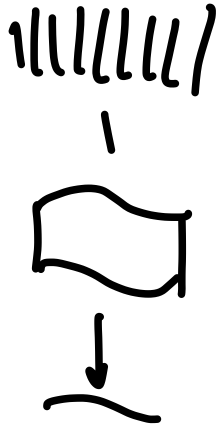
Take the flattening stratification for $\pi_*\mathcal{F}(m)$

→ no obvious common refinement.

First Lemma

$\mathcal{F}$ coherent (no flatness hypothesis)

TRADE-OFF

Situation: 

$$\begin{array}{c} \mathbb{P}^n_A \\ \downarrow \\ \operatorname{Spec} A \end{array}$$

Noetherian

"After you twist $\mathcal{F}$ enough, ~~cohomology~~ → higher pushforward commutes with one base change."

Precisely: Fix $\operatorname{Spec} B \to \operatorname{Spec} A$. (i.e., $B \leftarrow A$)

We have

$$\begin{tikzcd} \mathcal{F}_B \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\ \mathbb{P}^n_B \arrow[r, "\mu"] \arrow[d, "\beta"'] & \mathbb{P}^n_A \arrow[d, "\alpha"] \\ \operatorname{Spec} B \arrow[r, "\rho"'] & \operatorname{Spec} A \end{tikzcd}$$

There is some M so that for $m \geq M$, $\rho^* R^i \alpha_* \mathcal{F}(m) \longrightarrow R^i \beta_* \mu^* \mathcal{F}(m)$ is an isomorphism.

First Lemma Situation:

$$\begin{tikzcd} \mathbb{P}^n_B \arrow[r, "u"] \arrow[d, "\beta"'] & \mathbb{P}^n_A \arrow[d, "\alpha"] \\ \operatorname{Spec} B \arrow[r, "\rho"] & \operatorname{Spec} A \end{tikzcd}$$

$\mathcal{F}_B$ on $\mathbb{P}^n_B$, $\mathcal{F}$ on $\mathbb{P}^n_A$ coherent (no flatness hypothesis)

$\operatorname{Spec} A$ Noetherian

Then there is some M so that for $m \geq M$, $\rho^* R^i\alpha_* \mathcal{F}(m) \to R^i\beta_* \mu^*\mathcal{F}(m)$ is an isomorphism.

Proof.

Case $i > 0$: you do it. $0 \longrightarrow 0$

Case $i=0$

$$\begin{tikzcd} \mathbb{P}^n_B \arrow[r, "u"] \arrow[d, "\beta"'] & \mathbb{P}^n_A \arrow[d, "\alpha"] \\ \operatorname{Spec} B \arrow[r, "p"] & \operatorname{Spec} A \end{tikzcd}$$

$\mathcal{F}_B$ on $\mathbb{P}^n_B$, $\mathcal{F}$ on $\mathbb{P}^n_A$ coherent (no flatness hypothesis)

Noetherian

Now for $a \gg 0$, $\mathcal{F}(a)$ is generated by global sections

$\Rightarrow \bigoplus^{\text{finite}} \mathcal{O} \to \mathcal{F}(a) \to 0 \Rightarrow$

$\exists \quad \bigoplus^{\text{finite}} \mathcal{O}_{\mathbb{P}^n_A}(-a) \to \mathcal{F} \to 0$

$\mathcal{G}$ coherent, $\quad 0 \to \mathcal{G} \to \bigoplus \mathcal{O}_{\mathbb{P}^n_A}(-a) \to \mathcal{F} \to 0$

Same trick for $\mathcal{G}$: $\quad \bigoplus \mathcal{O}_{\mathbb{P}^n_A}(-b) \to \bigoplus \mathcal{O}_{\mathbb{P}^n_A}(-a) \to \mathcal{F} \to 0$

$$\begin{tikzcd} \mathbb{P}^n_B \arrow[r, "a"'] \arrow[d, "\beta"'] & \mathbb{P}^n_A \arrow[d, "\alpha"] \\ \operatorname{Spec} B \arrow[r, "\rho"'] & \operatorname{Spec} A \end{tikzcd}$$

$\mathcal{F}_B \qquad \mathcal{F}$

$$0 \to \mathcal{G} \to \bigoplus \mathcal{O}_{\mathbb{P}^n_A}(-a) \to \mathcal{F} \to 0$$

$$\bigoplus \mathcal{O}_{\mathbb{P}^n_A}(-b) \to \bigoplus \mathcal{O}_{\mathbb{P}^n_A}(-a) \to \mathcal{F} \to 0$$

$$0 \to \pi_* \mathcal{G} \to \pi_* \mathcal{O}_{\mathbb{P}^n}(-c) \to \pi_*(\mathcal{F})$$

$$\pi_* \mathcal{O}_{\mathbb{P}^n_*}(-b)$$

Pull back to B: (left-exact)

$$0 \to \mathcal{H} \to \bigoplus \mathcal{O}_{\mathbb{P}^n_B}(-a) \to \mathcal{F}_B \to 0$$

$$\bigoplus \mathcal{O}_{\mathbb{P}^n_B}(-b) \to \bigoplus \mathcal{O}_{\mathbb{P}^n_B}(-a) \to \mathcal{F}_B \to 0$$

From last page:

$$\bigoplus_{\mathbb{P}^n_A} \mathcal{O}(-b) \to \bigoplus_{\mathbb{P}^n_A} \mathcal{O}(-a) \to \mathcal{F} \to 0$$

Twist $(\otimes \mathcal{O}(m))$:

$$\bigoplus \mathcal{O}(-b+m) \to \bigoplus \mathcal{O}(-a+m) \to \mathcal{F}(m) \to 0$$

$$0 \to \mathcal{G}(m) \to \bigoplus \mathcal{O}(-a+m) \to \mathcal{F}(m) \to 0$$

Pullback (u^*):

$$\bigoplus_{\mathbb{P}^n_B} \mathcal{O}(-b+m) \to \bigoplus \mathcal{O}(-a+m) \to \mathcal{F}_B(m) \to 0$$

$$0 \to \mathcal{H}(m) \to \bigoplus \mathcal{O}(-a+m) \to \mathcal{F}_B(m) \to 0$$

$$\begin{array}{ccc} \mathcal{F}_B & & \mathcal{F} \\ \mathbb{P}^n_B & \xrightarrow{u} & \mathbb{P}^n_A \\ \beta \downarrow & & \downarrow \alpha \\ \operatorname{Spec} B & \xrightarrow{\rho} & \operatorname{Spec} A \end{array}$$

Push forward (α_*):

$$\alpha_* \bigoplus \mathcal{O}(-b+m) \to \alpha_* \bigoplus \mathcal{O}(-a+m) \to \alpha_* \mathcal{F}(m) \to R^1 \alpha_* \mathcal{G}(m)$$

$$\downarrow$$

$$0$$

$m \gg 0$

$$\beta_* \bigoplus \mathcal{O}(-b+m) \to \beta_* \bigoplus \mathcal{O}(-a+m) \to \beta_* \mathcal{F}_B(m) \to R^1 \beta_* \mathcal{H}(m)$$

$$R^1 \beta_* \mathcal{H}(m) \to 0$$

$$\begin{tikzcd} \mathcal{F}_B \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\ \mathbb{P}^n_B \arrow[r, "\mu"'] \arrow[d, "\beta"'] & \mathbb{P}^n_A \arrow[d, "\alpha"] \\ \operatorname{Spec} B \arrow[r, "\rho"'] & \operatorname{Spec} A \end{tikzcd}$$

Conclusion: for $m \gg 0$ (so $R^1\mathcal{H}(m) = 0$, $R^1\mathcal{G}(m) = 0$)

$$\begin{tikzcd} \bigoplus \rho^*\alpha_*\left(\mathcal{O}(-b)^{m}\right) \arrow[r] \arrow[d, "\sim"] & \bigoplus \rho^*\alpha_*\left(\mathcal{O}(-a)^{m}\right) \arrow[r] \arrow[d, "\sim"] & \boxed{\rho^*\alpha_*\mathcal{F}(m)} \arrow[r] \arrow[d, dotted, "\sim"] & 0 \arrow[d, "\sim"] \\ \bigoplus \beta_*\mu^*\left(\mathcal{O}(-b)^{m}\right) \arrow[r] & \bigoplus \beta_*\mu^*\left(\mathcal{O}(-a)^{m}\right) \arrow[r] & \boxed{\beta_*\mu^*\mathcal{F}(m)} \arrow[r] & 0 \end{tikzcd}$$

Then

by the five lemma we win. //

One more fact to recall (fairly serious cleverness): Generic flatness.

Situation:

$$\begin{array}{ll} & \mathcal{F} \quad \text{coherent} \\ \mathbb{P}^n_A & \\ \downarrow & \\ \operatorname{Spec} A & \text{Noetherian} \quad \text{(Reduced)} \ \boxed{\text{(Integral)}} \end{array}$$

BLACK BOX

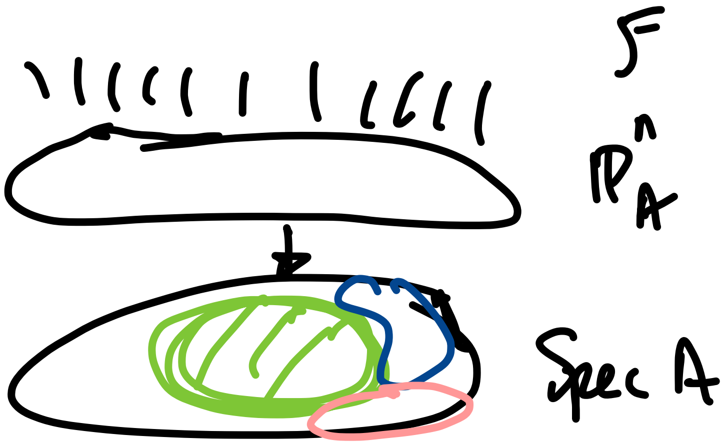
Then there is a dense open subset of Spec A over which $\mathcal{F}$ is flat.

We are ready to begin!

Proof of the Flattenning Stratification:

First step: locally closed subsets (even affine) | on which it is flat.

reduced structure.



What sucks:

not optimal; not a stratification; no scheme structure.

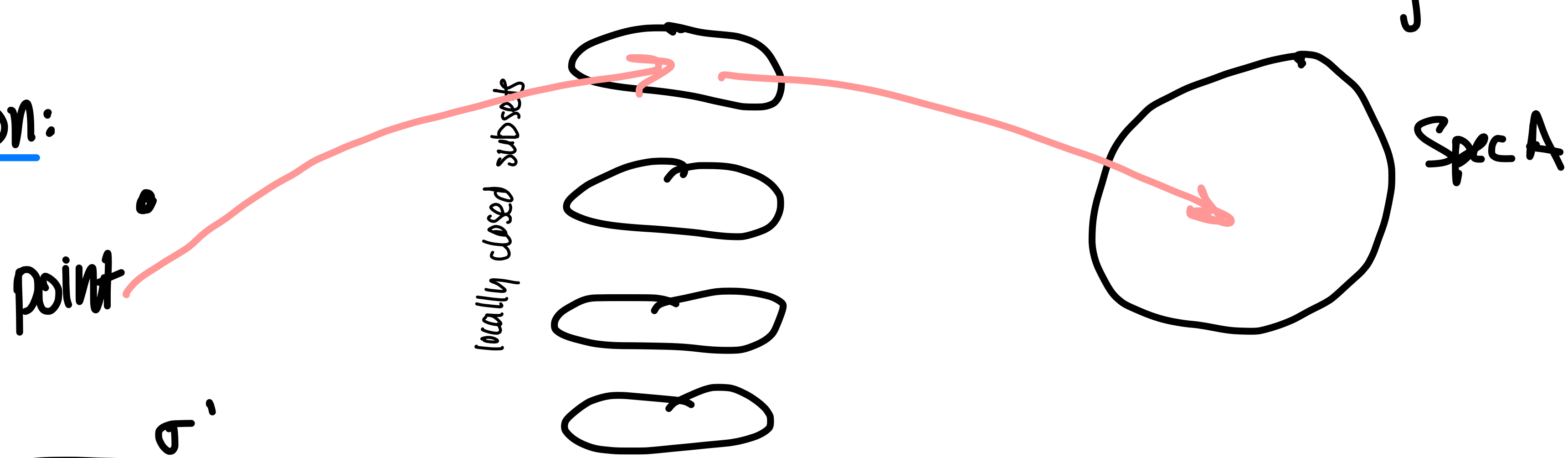
But: "finiteness" of some sort.

We now know only finitely many Hilbert polynomials are possible!

$(m \geq M)$

Second step: After a sufficiently big twist, cohomology commutes with base change to all points of Spec A.

Reason:



$$\begin{tikzcd}
\mathbb{P}^n_K \arrow[r, "\mu'"] \arrow[d, "\pi_K"] \arrow[rr, bend left, "\sigma'"] & \mathbb{P}^n_B \arrow[r, "\rho'"] \arrow[d, "\pi_B"] & \mathbb{P}^n_A \arrow[d, "\pi_A"] \\
\operatorname{Spec} K \arrow[r, "\mu"] \arrow[rr, bend right, "\sigma"'] & \operatorname{Spec} B \arrow[r, "\rho"] & \operatorname{Spec} A
\end{tikzcd}$$

$\mathbb{P}^n_K$ — $\mathcal{F}_K$, $\mathbb{P}^n_B$ — $\mathcal{F}_B$, $\mathbb{P}^n_A$ — $\mathcal{F}$

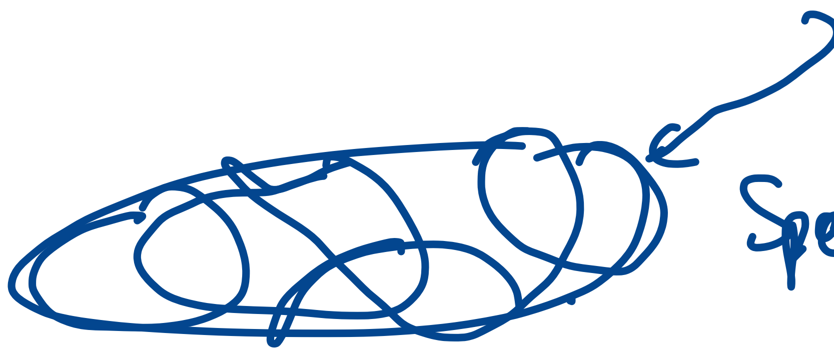
$$(\sigma)^* R^i(\pi_A)_* \mathcal{F}(m) \to R^i(\pi_K)_* (\sigma')^* \mathcal{F}(m)$$

Third step Let's get the candidate topological stratification!

The Hilbert polynomial of $\mathcal{F}$ for each point $p \in \operatorname{Spec} A$ is a polynomial of degree n. So it is determined by its values at $n+1$ integers.

Take rank of $\pi_* \mathcal{F}(M), \dots, \pi_* \mathcal{F}(M+n)$.

(Explain)

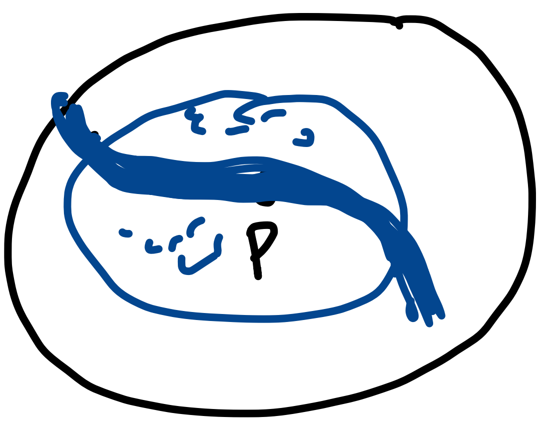


Spec A

We now have finitely many locally closed subsets on which the Hilbert polynomials are constant. Next...

Fourth step: get the (candidate) scheme structure on these strata.

(shrink so the strata is closed)



Spec A

$\pi_* \mathcal{F}(M)$, is a coherent sheaf on Spec A, hence by the flattenning stratification there is a maximal closed subscheme containing p on which it is locally free

Shrink Spec A so that: $I_0 \subset A$

Also for $\pi_* \mathcal{F}(M+1)$, $\pi_* \mathcal{F}(M+2)$, ..., $\pi_* \mathcal{F}(M+n)$.

I_1 $\quad$ I_2 $\quad$ I_n

$I_0 + I_1 + I_2 + \cdots + I_n$.

$$I_0 \subset I_0 + I_1 \subset I_0 + I_1 + I_2 \subset \cdots \subset I_0 + \cdots + I_n$$

Keep going: $\pi_* \mathcal{F}(M+i)$ $\quad i = n+1, n+2, \ldots$

$$\subset I_0 + \cdots + I_{n+1} \subset \cdots$$

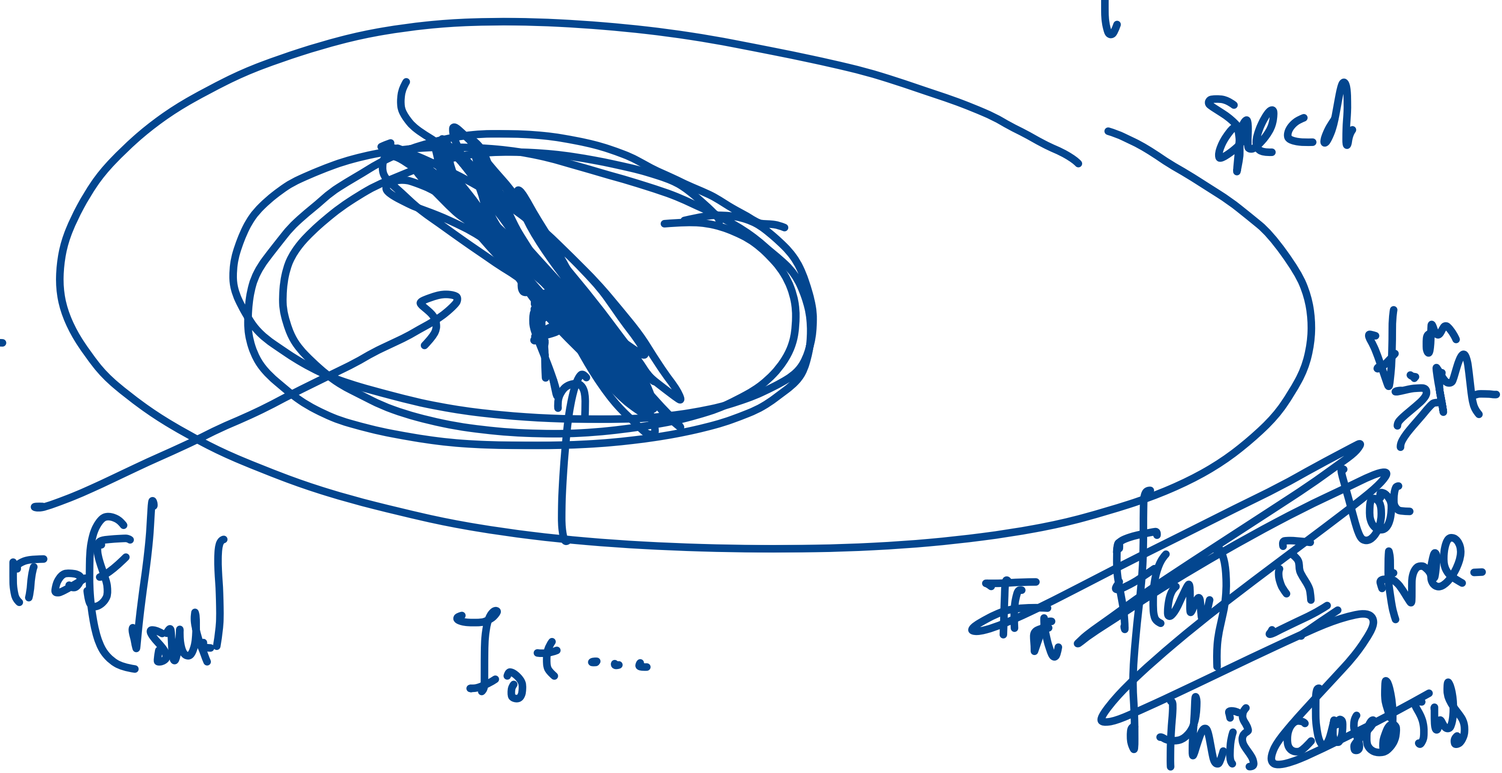
By Noetherianity, this eventually stabilizes!

Now:

$\pi_* \mathcal{F}(m)$ $m \geq M$.

subsheaf

It is loc free!!



Fifth step $\mathcal{F}$ is actually flat over this closed subscheme. Call this subscheme $\operatorname{Spec} B$.

$$\begin{tikzcd}
 & \mathcal{F}_B \arrow[dl, dash] & & \mathcal{F} \arrow[dl, dash] \\
\mathbb{P}^n_B \arrow[rr] \arrow[d, "\beta"'] & & \mathbb{P}^n_A \arrow[d] & \\
\operatorname{Spec} B \arrow[rr] & & \operatorname{Spec} A &
\end{tikzcd}$$

Now for this <u>fixed</u> $\operatorname{Spec} B$, then (higher) pushforwards commute with base change for $\mathcal{F}(m)$ for $m \geq M$ for some M. Thus $\beta_* \mathcal{F}_B(m)$ is locally free for $m \gg 0$, so (by our criterion) $\mathcal{F}_B$ is flat!

Sixth step

This (on our shrunken neighborhood of p) satisfies our universal property!

Proof: Suppose we have

$$\begin{tikzcd} \mathcal{F}_B \arrow[d, dash] & & \mathcal{F} \arrow[d, dash] \\ \mathbb{P}^n_B \arrow[rr, "\rho"] \arrow[d, "\beta"'] & & \mathbb{P}^n_A \arrow[d, "\alpha"] \\ \operatorname{Spec} B \arrow[rr, "\pi"] & & \operatorname{Spec} A \end{tikzcd}$$

such that $\mathcal{F}_B$ is flat over B.

For $m \gg 0$, $\pi^* \alpha_* \mathcal{F}(m) \longrightarrow \beta_* \rho^* \mathcal{F}(m)$ is an isomorphism (and higher pushforwards are zero) so (by the universal property of the flattening stratification of $\alpha_* \mathcal{F}(m)$) $\operatorname{Spec} B$ maps to the right stratum.

Seventh step (that you may not have noticed): all of these constructions glue together (by a universal property argument).

This completes the proof! (Do you agree?)

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