Moduli Spaces in Algebraic Geometry

Math 245 A (winter 2022)

Jan. 24, 2022.

Medium term plan.

Construction / Representability / Existence of the Hilbert scheme (and the Quot scheme)

Concepts:

the flattening stratification

cohomology and base change

cohomology vanishing via Castelnuovo-Mumford Regularity

LAST DAY:

Theorem (e.g. "The Rising Sea" notes)

$A^{\oplus m} \to A^{\oplus n} \to M \to 0$

(i) (algebra) Suppose M is a finitely presented module over a local ring A.

Then M is flat iff M is free iff M is projective

(opinion: in a precise sense, this is not hard.)

(ii) (geometry) Suppose $\mathcal{F}$ is a finitely presented (quasicoherent) sheaf on a scheme X. Then $\mathcal{F}$ is flat $/X$ iff $\mathcal{F}$ is locally free.

The flattenning stratification for finitely presented sheaves on any scheme

Theorem Suppose X is a scheme, and $\mathcal{F}$ is a finitely presented (q-coh) sheaf on X. Then there are (uniquely determined) locally closed subschemes $U_0, U_1, U_2, \ldots \hookrightarrow X$ such that for all $\pi: Y \to X$, $\pi^* \mathcal{F}$ is a rank n locally free sheaf (on Y) if and only if π factors through $U_n \hookrightarrow X$.
Moreover, $|U_0|, |U_1|, \ldots$ (the underlying locally closed subsets of X) form a (topological) stratification of $|X|$. (Reminder of proof strategy)

Why are Hilbert polynomials (locally constant in flat families)?

Set-up

$$
\begin{tikzcd}
 & \mathcal{F}_p \arrow[dl, dash] & & \mathcal{F} \arrow[dl, dash] \arrow[dr, dash] & \\
X_p \arrow[rr] \arrow[d] & & \cancel{X} \arrow[d] & \subset & \mathbb{P}^n_S \arrow[dll] \\
p & \in & S & &
\end{tikzcd}
$$

$\mathcal{F}$ coherent flat$/S$.

S Noetherian (for now?)

$m \in \mathbb{Z}$

$$\chi(\mathcal{F}_p(m)) = h^0(\mathcal{F}_p(m)) - h^1(\mathcal{F}_p(m)) \quad \cdots$$

is a polynomial in m (the Hilbert polynomial)

Looking closer:

$$
\begin{tikzcd}
& \mathcal{F}_p \arrow[dl, dash] & & \mathcal{F}_B \arrow[dl, dash] & & \mathcal{F} \arrow[dl, dash] \\
X_p \arrow[rr] \arrow[d] \arrow[drr, phantom, "\ulcorner", very near start] & & X_B \arrow[rr, hook] \arrow[d] \arrow[drr, phantom, "\ulcorner", very near start] & & X \arrow[d, "\pi"] & \text{projective} \\
P \arrow[rr] & & \operatorname{Spec} B \arrow[rr, hook, "\text{open}"] & & S &
\end{tikzcd}
$$

$h^i(X_p, \mathcal{F}_p)$ $\qquad$ $H^i(X_B, \mathcal{F}_B) = R^i\pi_*\mathcal{F}|_{\operatorname{Spec} B}$ $\qquad$ $R^i\pi_*\mathcal{F}$ coherent (harder to show if π is only proper)

relation?

nice relation

Translation:

How does cohomology behave under base change?

(discuss)

Let's investigate!

$$\begin{tikzcd}
\mathcal{F}_A \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\
X_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\substack{\text{projective} \Rightarrow \\ \text{quasicompact} \\ +\ \text{separated}}"] \\
\operatorname{Spec} A \arrow[r, "\text{affine morphism}"'] & \operatorname{Spec} B
\end{tikzcd}$$

Let $X = \bigcup U_i$ affine cover

Čech Complex for $\mathcal{F}$

$$\check{C}^\bullet : \quad 0 \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_i \cap U_j) \to \cdots \to \prod \mathcal{F}(U_1 \cap \cdots \cap U_n) \to 0$$

This a complex of B-modules.

$H^i(X, \mathcal{F})$ is the i^{th} cohomology of $\check{C}^\bullet$.

(B-modules)

Pull this back to Spec A, i.e., apply $\boxed{\otimes_B A.}$

We get a new Čech complex:

$\check{C}^\bullet \otimes_B A$: $\qquad 0 \to \prod \mathcal{F}(U_i) \otimes_B A \to \quad \cdots$

H^i of <u>this</u> complex is $H^i(X_A, \mathcal{F}_A)$. By the FHHF Theorem (where F is the functor $\cdot \otimes_B A$, which is right exact) we get natural maps

$$FH \longrightarrow HF$$

i.e. $\quad H^i(X, \mathcal{F}) \otimes_B A \longrightarrow H^i(X_A, \mathcal{F}_A)$.

If $\otimes_B A$ is exact — i.e. A is <u>flat</u>$/_B$ — then this is an iso!

For example:

$$\begin{tikzcd} \mathcal{F}_{\mathrm{Spec}\,\mathcal{O}_{S,p}} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\ X_{\mathrm{Spec}\,\mathcal{O}_{S,p}} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \mathrm{Spec}\,\mathcal{O}_{S,p} \arrow[r] & S \end{tikzcd}$$

$$H^i\left(X_{\mathrm{Spec}\,\mathcal{O}_{S,p}}, \mathcal{F}_{\mathrm{Spec}\,\mathcal{O}_{S,p}}\right) \equiv \left(R^i\pi_*\mathcal{F}\right)_p$$

"The cohomology of the germ is the germ of the cohomology."

Fun and useful fact

Situation:

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, dash] & \text{coherent} \\ X \arrow[d, "\pi \text{ proper}"] & & \\ S & & \end{tikzcd}$$

Noetherian S

If $\mathcal{F}$ is flat over S, and $R^i\pi_*\mathcal{F} = 0$ for $i>0$, then $\pi_*\mathcal{F}$ is a (finite rank) vector bundle (locally free sheaf).

Proof This is a local question, so we can reduce to the case where S is affine, $S = \operatorname{Spec} A$, say. The Čech complex

$$\check{C}^\bullet : \quad 0 \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_i \cap U_j) \to \cdots \to \prod \mathcal{F}(U_1 \cap \cdots \cap U_n) \to 0$$

is a complex of flat A-modules.

It is exact except at the left, where the kernel is $H^0(X, \mathcal{F})$. So

$$0 \to H^0(X,\mathcal{F}) \to \prod \mathcal{F}(U_i) \to \cdots \to \prod \mathcal{F}(U_1 \cap \cdots \cap U_n) \to 0$$

(the "extended Čech complex") is exact.

(Exercise: In a finite complex, if all but the first entry are flat, so is the the first entry. (Hint: break up into short exact sequences.))

Hence $H^0(X,\mathcal{F})$ is flat and coherent hence $\widetilde{H^0(X,\mathcal{F})}$ is locally free.

//

Furthermore: under <u>any</u> base change

$$\begin{tikzcd} \mathcal{F}_T \arrow[d, dash, dashed] & & \mathcal{F} \arrow[d, dash, dashed] \\ X_T \arrow[r] \arrow[d, "\rho"'] & X \arrow[d, "\pi"] & \\ T \arrow[r, "\alpha"'] & S & \end{tikzcd}$$

$R^i \rho_* \mathcal{F}_T = 0$ for $i > 0$

$\rho_* \mathcal{F}_T = \alpha^* \pi_* \mathcal{F}$ is finite rank locally free.

(in particular, coho commutes with base change)

<u>Proof</u> We may take S and T to be affine, say $\operatorname{Spec} A$ and $\operatorname{Spec} B$.

The extended Čech complex $0 \to H^0(X, \mathcal{F}) \to \check{C}^\bullet \to 0$ is a (bounded) exact sequence of flat A-modules, hence remains exact under <u>any</u> base change, so $0 \to H^0(X,\mathcal{F}) \otimes_A B \to \check{C}^\bullet \otimes_A B \to 0$ is exact, so we know the cohomology of $0 \to \check{C}^\bullet \otimes_A B \to 0$ //

Corollary Situation:

$$\begin{array}{ccl} & & \text{flat,} \\ \mathcal{F} & & \text{coherent} \\ & X & \\ \text{proper} & \downarrow & \pi \\ & S & \text{loc. Noeth.} \end{array}$$

$$\boxed{H^0(X_p, \mathcal{F}_p) = (\pi_* \mathcal{F})\big|_p}$$

$\pi_* \mathcal{F}$ is a vec. bundle.

If $R^i \pi_* \mathcal{F} = 0$ for $i > 0$, then $h^0(X_p, \mathcal{F}_p)$ is a locally constant function of S.

Proof It is the rank at $p \in S$ of a vector bundle on S. //

HILBERT POLYNOMIALS

Proposition Situation:

$$\begin{array}{c} \mathcal{F} \searrow \\ \mathbb{P}^n_k \\ \downarrow \\ \operatorname{Spec} k \end{array}$$

coherent

Then $\chi(\mathcal{F}(m))$ is a polynomial in m of degree at most n.

Proof by induction on n.

$n=0$: $\chi(\mathcal{F}(m)) = \chi(\mathcal{F}) = h^0(\mathcal{F})$ is constant.

Inductive step: choose a hyperplane $\mathbb{P}^{n-1} \cong H \subset \mathbb{P}^n$.
with defining equation $x \in H^0(\mathbb{P}^n, \mathcal{O}(1))$.

Then $0 \to A \to F \xrightarrow{\times x} F(1) \to B \to 0$

supported on H

Let $p(m) = \chi(F(m))$.

We have: $0 \to A(m) \to F(m) \to F(m+1) \to B(m) \to 0$

$$\Rightarrow \quad p(m+1) - p(m) = \underbrace{\chi(B(m)) - \chi(A(m))}_{\text{polynomials of degree } \leq n-1 \text{ by inductive hypothesis.}}$$

$\therefore$ $p(m)$ is a polynomial of degree $\leq n$. //

Theorem Hilbert polynomials are locally constant in flat families.

Proof Consider
$$\begin{array}{ccl} \mathcal{F} & & \text{flat coherent.} \\ & \searrow & \\ & \mathbb{P}^n_A & \\ & \downarrow \pi & \\ & \operatorname{Spec} A & \text{Noetherian} \end{array}$$

By Serre vanishing, there is some N so that $H^i(\mathbb{P}^n_A, \mathcal{F}(m)) = 0$ for $m > N$. Then...

$R^i\pi_* \mathcal{F}(m) = 0$ for $i > 0$. $\pi_* \mathcal{F}(m)$ is a vector bundle.

Then: over any $p \in \operatorname{Spec} A$, $h^i(\mathcal{F}(m)|_p) = 0$ $i > 0$

$h^0(\mathcal{F}(m)|_p)$ loc. const.

$\chi(m)$ loc. constant.

Theorem Situation: $\mathcal{F} \curvearrowright \mathbb{P}^n$ coherent

$$\begin{array}{c} \mathbb{P}^n \\ | \\ S := \operatorname{Spec} A \end{array}$$

Noetherian

Then $\mathcal{F}$ is flat over S if ~~near every point p of S,~~ there is $N > 0$ such that

$\pi_* \mathcal{F}(m)$ is a finite rank vector bundle ~~in that neighborhood~~ for $m > N$.

Proof One direction: we just did.

Other direction? $\mathbb{P}^n = \bigcup_{i=0}^{n} U_i \hookrightarrow D(x_i)$

$\tilde{M} \cong \mathcal{F}$

over U_i, $\mathcal{F}(U_i)$?

$M = \bigoplus_{m > N} H^0(\mathcal{F}(m))$ graded module over $A[x_0, \dots, x_n]$. flat / A.

$(M_{x_i})_{\text{degree } 0}$ piece

flat? $\hookleftarrow$ tot. flat

Next, let's try to generalize our fun fact:

Fun and useful fact

Situation:

$$\begin{array}{ccc} & & \mathcal{F} \text{ coherent} \\ & \diagup & \\ X & & \\ \downarrow \pi & \text{proper} & \\ S & & \end{array}$$

Noetherian S

If $\mathcal{F}$ is flat over S, and $R^i \pi_* \mathcal{F} = 0$ for $i>0$, then $\pi_* \mathcal{F}$ is a (finite rank) vector bundle

Idea:

$$\begin{array}{ccc} & & \mathcal{F} \text{ coherent flat} \\ & \diagup & \\ X & & \\ \downarrow \pi & \text{proper} & \\ \operatorname{Spec} A = S & & \end{array}$$

Noetherian

$$\check{C}^\bullet = 0 \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_i \cap U_j) \to \cdots \to \prod \mathcal{F}(U_1 \cap \cdots \cap U_n) \to 0$$

complex of flats.

Cohomology is coherent.

More generally:

We will construct a complex of finitely-generated free A-modules

$$\cdots \to A^{\oplus} \to A^{\oplus} \to A^{\oplus} \to 0$$

(i) Where the cohomology is $H^i(X, \mathcal{F})$.

↳ in a strong sense.

② Including after any pullback!

$$\begin{tikzcd} \mathcal{F}_B \arrow[dr, dash] & & \mathcal{F} \arrow[dr, dash] & \\ & X_B \arrow[rr] \arrow[d, "\pi_B"'] \arrow[drr, phantom, "\ulcorner", very near start] & & X \arrow[d, "\pi"] \\ & \operatorname{Spec} B \arrow[rr] & & \operatorname{Spec} A \end{tikzcd}$$

$R^i(\pi_B)_* \mathcal{F}_B$ is identified with the i^{th} cohomology of the pulled back complex!

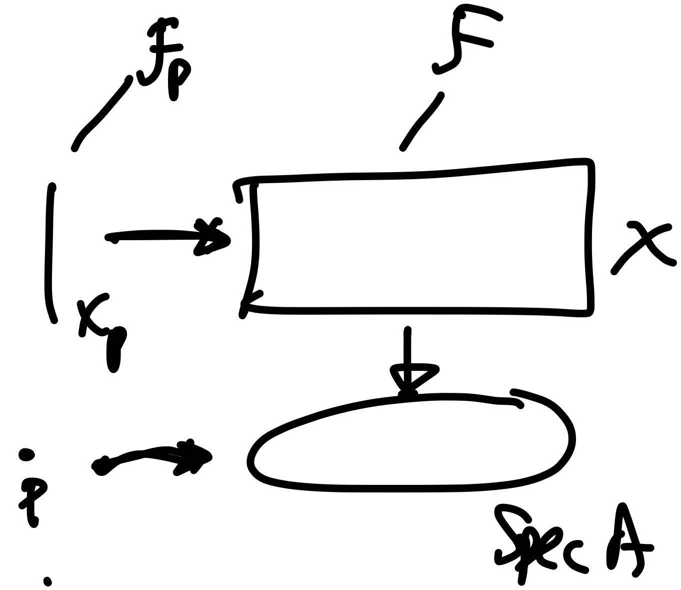
Already useful:

Theorem

Uppersemicontinuity of cohomology groups in flat families ✓

Proof Consider i^{th} cohomology group.

$$\cdots \to A^{\oplus n_{i-1}} \xrightarrow[\in A]{\text{matrices}} A^{\oplus n_i} \xrightarrow[\in A]{\text{matrices}} A^{\oplus n_{i+1}} \to \cdots$$



$h^i(X_p)$ vary with p.

"Mumford complex"

dim of homology here: rank of a matrix

(dim of image) is lower-semicontinuous. $\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$ cheap!

$$\xrightarrow[\alpha]{\text{rank } a} \overset{\text{dim } b}{V} \xrightarrow[\gamma]{\text{rank } c}$$

dim of coho here?

$$= \dim \ker\gamma - \dim \operatorname{im}\alpha$$
$$= b - \dim \operatorname{im}\gamma - \dim \operatorname{im}\alpha$$
$$= b - \operatorname{rank}\gamma - \operatorname{rank}\alpha. \quad \Big\} \to \text{uppersemicts.}$$

constant.