Moduli Spaces in Algebraic Geometry

Math 245 A (Winter 2022)

Jan. 21, 2022.

Where we are:

We are pondering the representability/construction of the Hilbert scheme

Define the Hilbert FUNCTOR for $\mathbb{P}^n$ as:

$$\mathrm{Hilb}_{\mathbb{P}^n} : \text{scheme } B \rightsquigarrow \begin{array}{ccc} X & \xrightarrow[\text{sub}]{\text{cl.}} & \mathbb{P}^n_B \\ {\scriptstyle \text{flat, f.pr.}} \searrow & & \swarrow \\ & B & \end{array}$$

brief discussion on hypotheses

Theorem (→ in process) $\mathcal{H}ilb_{\mathbb{P}^n}$ is representable, by a scheme we call the Hilbert scheme of $\mathbb{P}^n$, $\mathrm{Hilb}\,\mathbb{P}^n$.

It suffices to show the representability of

$$\mathcal{H}ilb^{p(t)\in\mathbb{Q}[t]}_{\mathbb{P}^n}$$

↑ Hilbert polynomial

This week:

flatness,

and the flattening stratification.

Some motivation

Suppose ${}^{\mathcal{F}}\searrow_{X}$ is a coherent sheaf on a Noetherian scheme.

careful. (drop this)

A slight generalization of a finite rank ~~vector bundle~~ → (locally free sheaf) on X.

Idea (basically proved soon): if X is reduced, then $\mathcal{F}$ is "mostly" a vector bundle.

(discuss)

On Spec A: $\quad \mathcal{F} = \widetilde{M}$ (M fin. generated A-module)

ψ

$p = [\mathfrak{p}]$

$\operatorname{rank}_p \mathcal{F} := \dim_{k(\mathfrak{p})} M_{\mathfrak{p}} / \mathfrak{p} M_{\mathfrak{p}}$

$k(\mathfrak{p}) := A_{\mathfrak{p}} / \mathfrak{p} A_{\mathfrak{p}}$.

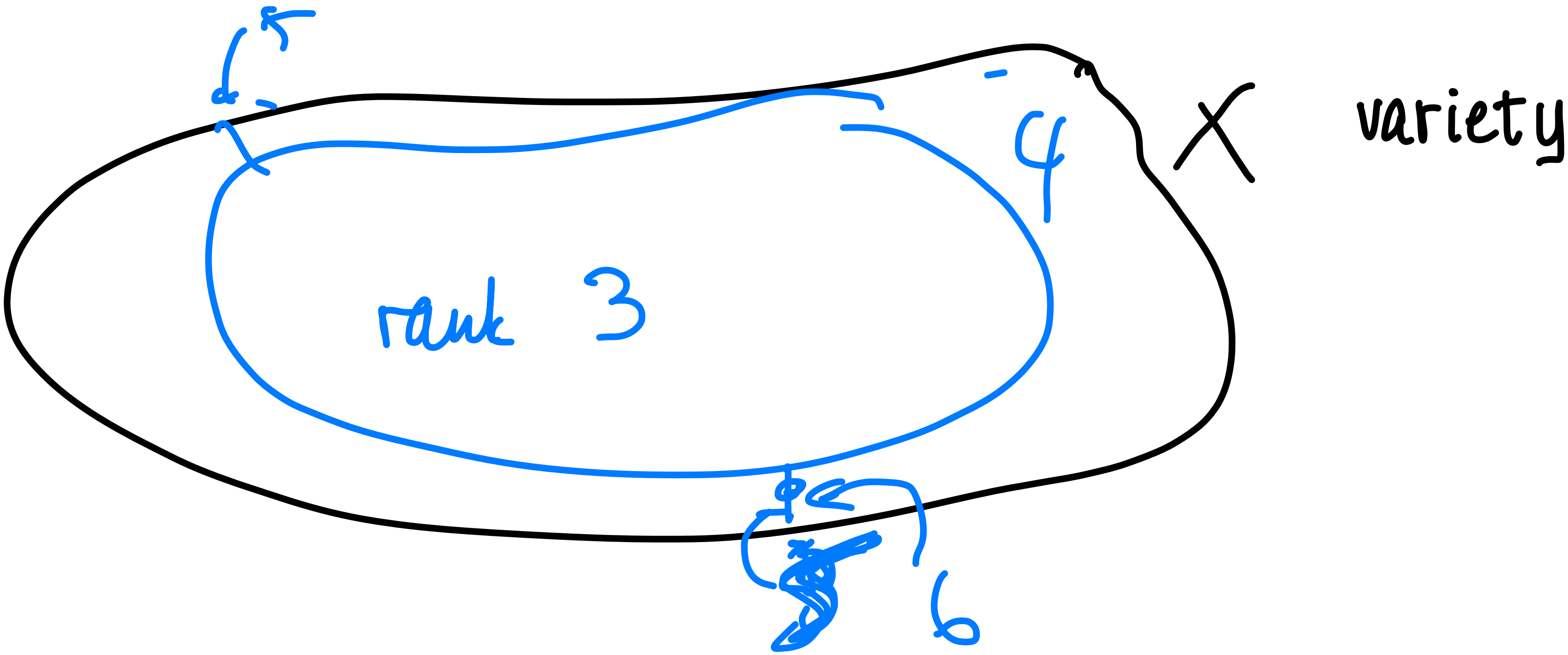
$\operatorname{rank} \mathcal{F} : X \to \mathbb{Z}^{\geq 0}$ is uppersemicontinuous (Pf: soon)

("jumps up" on closed subsets)

On the big open set where the rank is minimal constant, $\mathcal{F}$ is a (finite rank) vector bundle (of that rank)

(references include: "ask me"; "The Rising Sea" notes; etc.)

∴ stratification of X by rank F.



Enter Flatness

Finite rank vector bundles are flat, i.e. basically, $\mathcal{E} \otimes$. is exact.

Reason: $\mathcal{O}^{\oplus n} \otimes \mathcal{F}$ is obviously exact.

$= \mathcal{F}^{\oplus n}$

$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$

$0 \to \mathcal{F}^{\oplus n} \to \mathcal{G}^{\oplus n} \to \mathcal{H}^{\oplus n} \to 0$

Theorem (e.g. "The Rising Sea" notes)

$A^{\oplus m} \to A^{\oplus n} \to M \to 0$

(i) (algebra) Suppose M is a finitely presented module over a local ring A. ~~Noetherian?~~

Then M is flat iff M is free iff M is projective

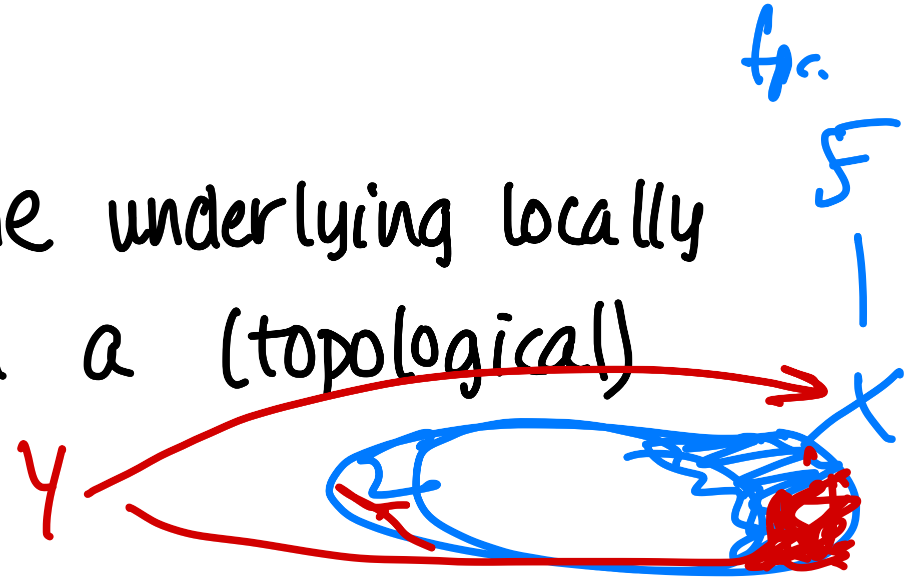
(opinion: in a precise sense, this is not hard.)

(ii) (geometry) Suppose $\mathcal{F}$ is a finitely presented (quasicoherent) sheaf on a scheme X. Then $\mathcal{F}$ is flat $/X$ iff $\mathcal{F}$ is locally free.

The flattenning stratification for finitely presented sheaves on any scheme

Theorem Suppose X is a scheme, and $\mathcal{F}$ is a finitely presented (q-coh) sheaf on X. Then there are (uniquely determined) locally closed subschemes $U_0, U_1, U_2, \ldots \hookrightarrow X$ such that for all $\pi: Y \to X$, $\pi^* \mathcal{F}$ is a rank n locally free sheaf (on Y) if and only if π factors through $U_n \hookrightarrow X$.

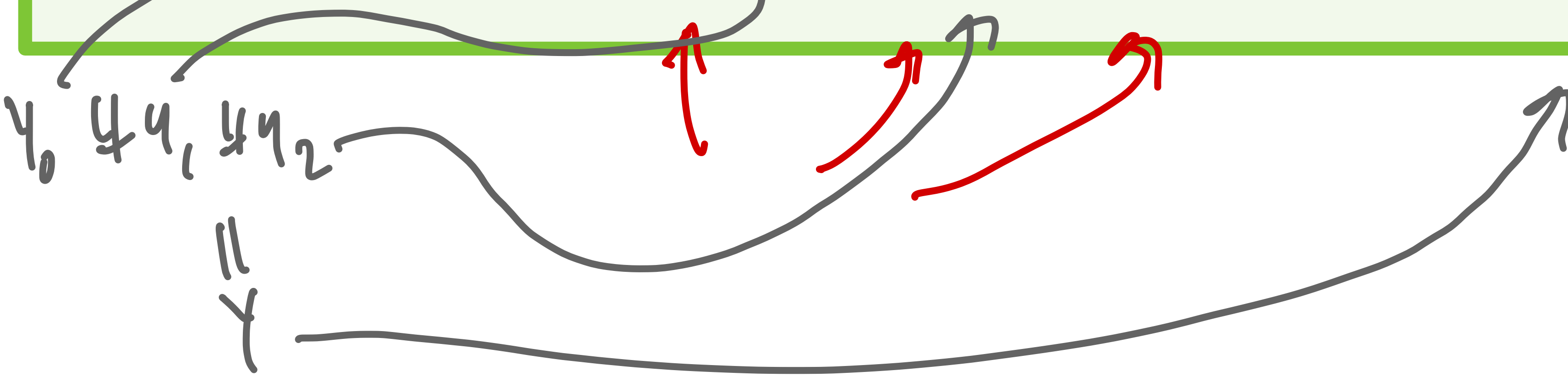
Moreover, $|U_0|, |U_1|, \ldots$ (the underlying locally closed subsets of X) form a (topological) stratification of $|X|$.



Striking consequence / formulation (?)

Given $\pi : Y \to X$, $\pi^* \mathcal{F}$ is a vector bundle if and only if π factors through

$$U_0 \amalg U_1 \amalg U_2 \amalg \cdots \longrightarrow X.$$



Proof

Strategy: First get the topological stratification $|U_0|, |U_1|, \ldots$

Then figure out the (locally closed sub)scheme structure

Then show it satisfies the universal property.

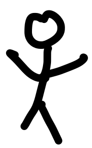
For the topological stratification: by rank.

$|U_n|$ locus where $\text{rank}_p \mathcal{F} = n$

Remark (may be worth discussing)

rank is preserved by pullback.

Reason: dim of a vector space is preserved by field extension.



$\tilde{M} = \mathcal{F}$

$[\mathfrak{q}] = q \in \operatorname{Spec} B \longrightarrow \operatorname{Spec} A \ni p = [\mathfrak{p}]$

$Y \subseteq \operatorname{Spec} B$, $X \subseteq \operatorname{Spec} A$, $Y \longrightarrow X$

$$(M \otimes_A k(p)) \otimes_{k(p)} k(q) \equiv (M \otimes_A B) \otimes_B k(q)$$

$$\begin{array}{ccc} M \otimes_A B & & M \\ | & & | \\ B & \longleftarrow & A \end{array}$$

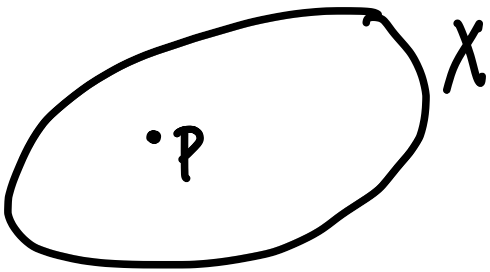
$M_\mathfrak{p} / \mathfrak{p} M_\mathfrak{p} =$ f.dim $k(\mathfrak{p})$ v.s

$A_\mathfrak{p} / \mathfrak{p} A_\mathfrak{p} = k(\mathfrak{p})$

$B_\mathfrak{q} / \mathfrak{q} B_\mathfrak{q} = k(\mathfrak{q})$

field extension: $A_\mathfrak{p} / \mathfrak{p} A_\mathfrak{p} \longrightarrow B_\mathfrak{q} / \mathfrak{q} B_\mathfrak{q}$

Next: the scheme structure near $p \in X$. ($\in U_n$.)



local ring ($\mathcal{O}_{X,p}$, maximal ideal m)

so $\kappa(p) = \mathcal{O}_{X,p}/m$ is the residue field

Let $n = \mathrm{rank}_p \mathcal{F}$, i.e.,

$\mathcal{F}_p / m\mathcal{F}_p$ is a dimension n vector space over the field $\kappa(p)$

$$\left(\mathcal{O}_{X,p} / m\mathcal{O}_{X,p}\right)^{\oplus n} \xrightarrow{\sim} \mathcal{F}_p / m\mathcal{F}_p$$

(copied on next page)

$$0 \to \left(\mathcal{O}_{X,p}/\mathfrak{m}\mathcal{O}_{X,p}\right)^{\oplus n} \xrightarrow{\sim} \mathcal{F}_p/\mathfrak{m}\mathcal{F}_p \to 0$$

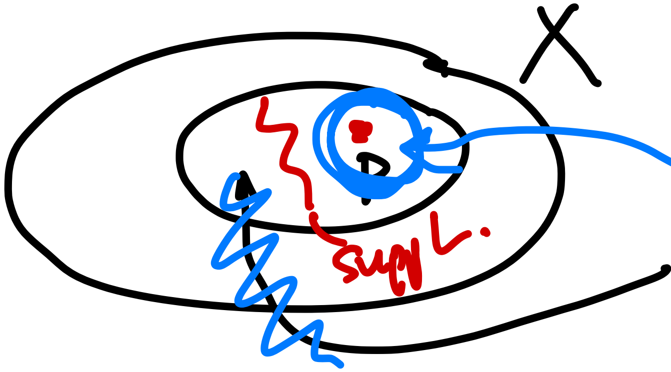
Lift to n elements of $\mathcal{F}_p$: $\quad \mathcal{O}_{X,p}^{\oplus n} \to \mathcal{F}_p$

This remains surjective:

$$\mathcal{O}_{X,p}^{\oplus n} \to \mathcal{F}_p \to 0$$

Reason: (Why?!) Nakayama!

These n germs of $\mathcal{F}$ "are" (lift to) actual sections of $\mathcal{F}$ in some small affine neighborhood of p.



$\mathcal{F}|_{\operatorname{Spec} A} = \widetilde{M}$

M finitely gen.

$\mathcal{F}|_{\operatorname{Spec} A}$

We have $A^{\oplus n} \to M$

may not be surjective.

$A^{\oplus n} \to M \to L \to 0$

f.g. A-module.

Supp L closed
Doesn't include p.

$A^{\oplus n} \to M \to 0$

f.pr.: $A^{\oplus ?} \to A^{\oplus ?} \to M \to 0$

$$0 \to K \to A^{\oplus n} \to M \to 0$$

K is finitely generated.

("Finitely presented implies always finitely presented" — great exercise)

Upshot: $A^{\oplus m} \longrightarrow A^{\oplus n} \longrightarrow M \longrightarrow 0$ on Spec A.

$\underbrace{\left[\text{matrix entries in } A\right]}_{m} \Big\} n$

Upshot: $A^{\oplus m} \to A^{\oplus n} \longrightarrow M \longrightarrow 0$ on Spec A.

[matrix entries in A]

How can M be locally free of rank n ?

For which A/I can M be locally free of rank n?

entries in I! ⊗ A/I

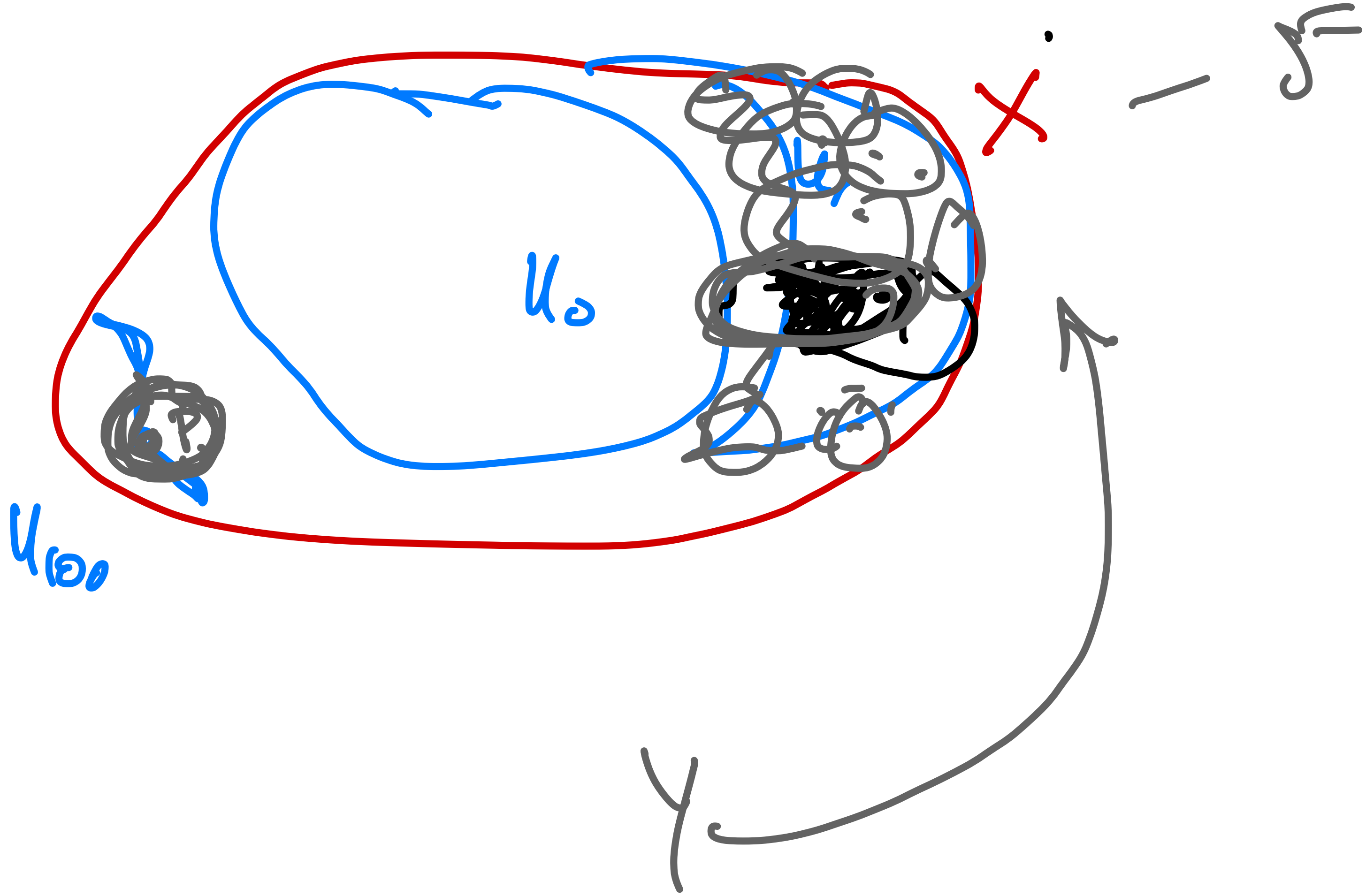
$(A/I)^{\oplus m} \longrightarrow (A/I)^{\oplus n} \longrightarrow M/IM \to 0$

[ ] ↪ 0 matrix

V(entries of matrix).

For which B → A can $M \otimes_A B$ be a free B-module of rank n?

Conclusion:



We have proved (modulo checking some details) The existence of the flattenning stratification for finitely presented sheaves on any scheme.

QED

Theorem (Flattening stratification in the full generality we will need)

Given:

$$\begin{array}{ccc} \mathbb{P}^n_X & - & \mathcal{F} \text{ finitely presented sheaf} \\ | & & \\ X & & \text{arbitrary scheme} \end{array}$$

Then there is a (canonical) stratification of X by locally closed subschemes of X, indexed by Hilbert polynomial, $U_{p(t)} \hookrightarrow X$ where for any $\pi: Y \to X$, $\pi^*\mathcal{F} - \mathbb{P}^n_Y \to Y$ is flat over Y with fibers having Hilbert polynomial $p(t)$ if and only if π factors through $U_{p(t)} \hookrightarrow X$.

Upshot, if you prefer: $\pi^*\mathcal{F}$ is flat over Y iff π factors through $\coprod_{p(t)} U_{p(t)} \longrightarrow X$.

Example: If $n=0$, this is the theorem we have just proved.