Moduli Spaces in Algebraic Geometry

Math 245A (Winter 2022)

Jan. 14, 2022.

No class Monday

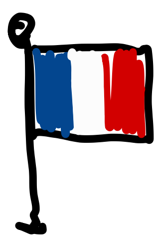
(Martin Luther King day holiday)

Where we are:

We are pondering the representability/construction of the Hilbert scheme

Define the Hilbert FUNCTOR for $\mathbb{P}^n$ as:

$$\mathrm{Hilb}_{\mathbb{P}^n} : \text{scheme } B \rightsquigarrow \begin{tikzcd} X \arrow[r, hook, "\text{cl.}", "\text{sub}"'] \arrow[dr, "\boxed{\text{flat, f.pr}}"'] & \mathbb{P}^n_B \arrow[d] \\ & B \end{tikzcd}$$



fppf = faithfully plat presentation finie

Theorem (another day) $\mathrm{Hilb}_{\mathbb{P}^n}$ is representable, by a scheme we call the Hilbert scheme of $\mathbb{P}^n$, $\mathrm{Hilb}\,\mathbb{P}^n$.

Last day: it suffices to show the representability of

$$\underset{\substack{p(t)\in\mathbb{Q}[t] \\ \text{Hilbert polynomial}}}{\mathrm{Hilb}}\ \mathbb{P}^n : \quad B \rightsquigarrow \begin{tikzcd} X \arrow[r, hook, "\text{cl.}"] \arrow[d, "p", "\text{fppf}"'] & \mathbb{P}^n_B \arrow[dl] \\ B & \end{tikzcd}$$

(geometric) fibers of p all have Hilbert pol. $p(t)$

next p: why "geometric" doesn't matter.

$$
\begin{tikzcd}
X_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \subset \mathbb{P}^n_k \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} k
\end{tikzcd}
$$

$K = \overline{K}$

Hilb pol of X

$\chi(X, \mathcal{O}(n))$

Today: two cases, hypersurfaces and linear spaces.

Hypersurfaces.

Fix degree d, ambient space $\mathbb{P}^n$.

Choice of FUNCTOR

① Parametrize things with that Hilbert polynomial. alg geo lmao.

(EX.) what you think.

② versions:

- me $\quad \begin{array}{c} \mathcal{X} \subset \mathbb{P}^n_B \\ \downarrow \\ B \end{array}$ fibers are cut out by degree d eqn.

Goal: Find $\mathcal{L}$, then $\Gamma(B, \mathcal{L})$ $a_{x_0^d}, \ldots,$

- geometric me geometric fiber.
- Spencer - Ben
- Ben // (if you care) $\quad \mathbb{P}^N$

Spencer-Ben FUNCTOR.

$$\begin{array}{ccc} B & \rightsquigarrow & \begin{tikzcd} X \arrow[r, hook, "d", "\text{sub}"'] \arrow[d] & \mathbb{P}^n_B \arrow[dl] \\ B & \end{tikzcd} \end{array}$$

such that...

$\boxed{\text{there exists}}$ affine cover of

$B = \bigcup \operatorname{Spec} R_i$

such that

$$\begin{array}{c} X_R \\ \downarrow \\ \operatorname{Spec} R \end{array}$$

$X_R = V(? x_0^d + \cdots + ? x_n^d)$

$\in R$

$\forall$ point of Spec R

pol is not zero.

Ben FUNCTOR.

$$\begin{tikzcd} X \arrow[r, hook, "cl."] \arrow[d] & \mathbb{P}^n_B \arrow[dl] \\ B & \end{tikzcd}$$

$\mathcal{L}$ line bundle on B.

$B \longmapsto$

$X \subset V(a'_{x_0^d} x_0^d + \cdots + a'_{x_n^d} x_n^d) \subset \mathbb{P}^n_B.$

$a_{x} \in \Gamma(B, \mathcal{L})$ $\quad$ $a'_{x} = \rho^* a_{x}$

At any pt of Spec B, a_{x} not all zero.

Spencer-Ben $\longleftrightarrow$ Ben.

Aside: $\mathcal{L} = \pi^* \mathcal{O}(1)$, $\pi^* y_0, \dots, \pi^* y_N$.

$$\begin{array}{ccc} & & \mathcal{O}(1) \quad y_0, \dots, y_N \\ & & | \\ B & \xrightarrow{\pi} & \mathbb{P}^N \end{array}$$

is same as

line bundle

$$\begin{array}{c} \mathcal{L} \\ | \\ B \end{array}$$

and $N+1$ sections of $\mathcal{L}$ not all zero at any point of B.

Our last **FUNCTOR**: expecting it to parametrized by

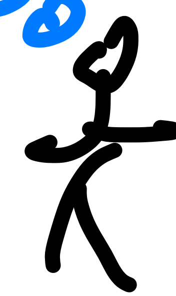
$$\begin{tikzcd} \overbrace{V(a_{x_0^d} x_0^d + \dots + a_{x_n^d} x_n^d)} = X \arrow[r, hook] \arrow[d] & \mathbb{P}^N \times \mathbb{P}^n \arrow[dl] \\ \left[a_{x_0^d}, \dots, a_{x_n^d}\right] \in \mathbb{P}^N & \end{tikzcd}$$

$N = \binom{n+d}{d} - 1$

$a_* \in \Gamma(\mathcal{O}_{\mathbb{P}^N}(1))$

$x_0 \in \Gamma(\mathcal{O}_{\mathbb{P}^n}(1))$

$\in \Gamma(\mathbb{P}^N \times \mathbb{P}^n, \mathcal{O}(1,d))$



$$\begin{tikzcd} & \hookrightarrow B \times \mathbb{P}^n \\ \arrow[u] B \arrow[r, "\pi"] & \mathbb{P}^N \end{tikzcd}$$

$B \leadsto V(a_{x_0^d} x_0^d + \dots + a_{x_n^d} x_n^d)$

i.e. $\begin{matrix} X \\ \downarrow \\ B \end{matrix}$ and $a_{x_0^n} \in \Gamma(B, \mathcal{L}), \dots, a_{x_n^n} \in \Gamma(B, \mathcal{L})$. ← not all zero @ any pt.

relative effective Cartier divisor:

Aside

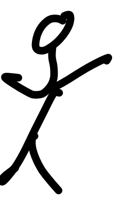
$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$

flat flat flat/S

english french

nice $\equiv$ plat

The local criterion for flatness.



$0 \to ? \to ? \to 0$

$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$

flat

$X \longrightarrow B$

Universal family.

$$\begin{array}{ccc} \mathcal{U} & \hookrightarrow & \mathbb{P}^N \times \mathbb{P}^n \\ \downarrow & & \\ \mathbb{P}^N & & \end{array} \qquad N = \binom{n+d}{d} - 1$$

$$[a_{x_0^d}, \ldots, a_{x_n^d}]$$

universal hypersurface

$$a_{x_0^d} x_0^d + \cdots + a_{x_n^d} x^d \in \Gamma/\mathcal{O}_{\mathbb{P}^N \times \mathbb{P}^n}(1,d))$$

(is it fppf?)

Pull it back.

$$B \longrightarrow \mathbb{P}^N$$

$$\mathcal{L} = \pi^* \mathcal{O}_{\mathbb{P}^N}(1). \qquad \pi^* a_m \in \Gamma(B, \mathcal{L})$$

$$\pi^* a_{x_0^d} x_0^d + \cdots + \pi^* a_{x_n^d} x_n^d \in \Gamma(\mathbb{P}^n_B, \rho^* \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}}(1))$$

Let's connect more FUNCTORS to $\mathrm{Hom}(\cdot, \mathbb{P}^N)$.

$$\begin{tikzcd}
\mathcal{X} \arrow[r, hook] \arrow[d] & \mathbb{P}^n_B \arrow[dl] \\
B &
\end{tikzcd}$$

$\mathcal{O}_{\mathbb{P}^n}(\mathcal{X})$ $\quad$ $\mathcal{O}(1)$ $\mathbb{P}^n$

such that blah-blah-blah.

(Goal: $\mathcal{L}$: $a_{x_0} \dots a_{x_i}$:) $\longrightarrow \mathbb{P}^N$:).

How do we find $\mathcal{L}$?

I will tell you: $\exists \mathcal{L}$, $a_{x_0^d}, \dots, a_{x_n^d} \in \Gamma(B, \mathcal{L})$.

$$\mathcal{X} = V(a_{x_0^d} x_0^d + \dots + a_{x_n^d} x_n^d).$$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"'] & \mathbb{P}^n_B \arrow[dl, "\pi"] \\ B & \end{tikzcd}$$

$\mathcal{O}(1)$

$X = V(s)$

$s \in \mathcal{O}_{\mathbb{P}^n_B}(\pi^*\mathcal{L} \otimes \mathcal{O}(d))$

$$\boxed{\text{Know } \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^n_B}(d) \text{ on } \mathbb{P}^n_B}$$

$$\pi_*\left(\mathcal{O}(X) \otimes \mathcal{O}(-d)\right) = \pi_*(\pi^*\mathcal{L})$$

What are the N+1 sections of $\mathcal{L}$. $a_{x^?}, \dots, a_{n^?}$