

Moduli Spaces in Algebraic Geometry

Math 245 A (winter 2022)

Jan. 12, 2022.

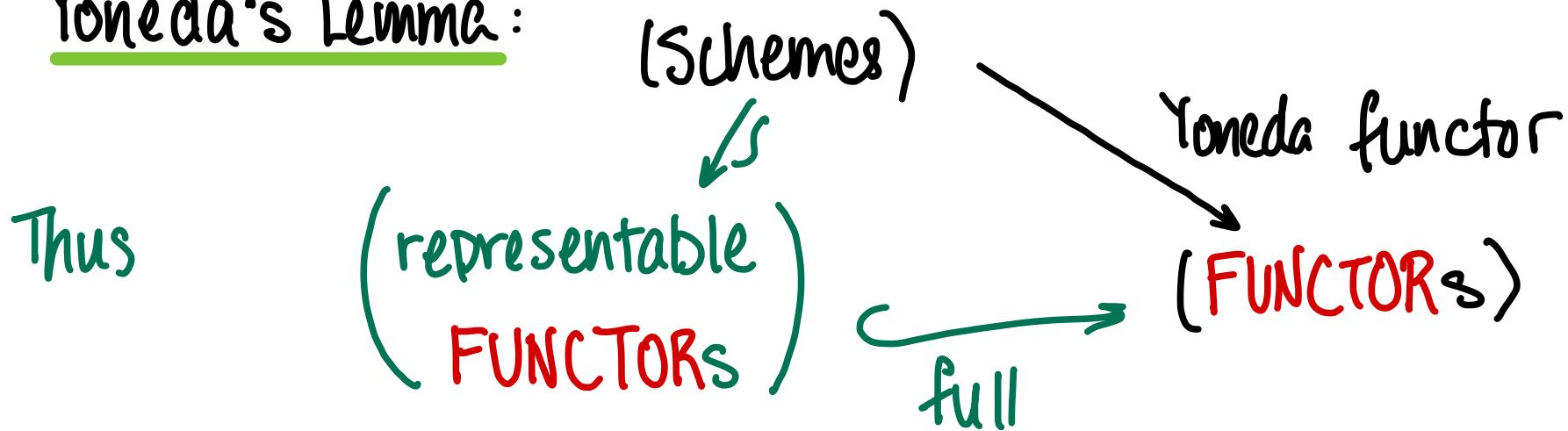
Where we are:

moduli space

existence =
construction =
representability

contravariant **FUNCTOR** from (Schemes) to (Sets)

Yoneda's Lemma:



Exercise in definitions: If X is a scheme and F is a **FUNCTOR**, then the maps $X \rightarrow F$ "are" the elements of $F(X)$. If $\alpha \in F(X)$, I immediately write $X \xrightarrow{\alpha} F$.

(Schemes) $\overset{\text{full}}{\subset}$ (SHEAVES) $\overset{\text{full}}{\subset}$ (FUNCTORS)

in the guise of
representable **FUNCTORS**



Definition:

A **FUNCTOR** \mathcal{F} is a **SHEAF** (in the Zariski topology) if for all Schemes X , the presheaf on X

$$U \subset X \mapsto \mathcal{F}(U)$$

is a sheaf.

Idea: we think of schemes as fundamentally geometric objects. **FUNCTORS** (especially moduli **FUNCTORS**) and **SHEAVES** often want to be thought of as geometric.

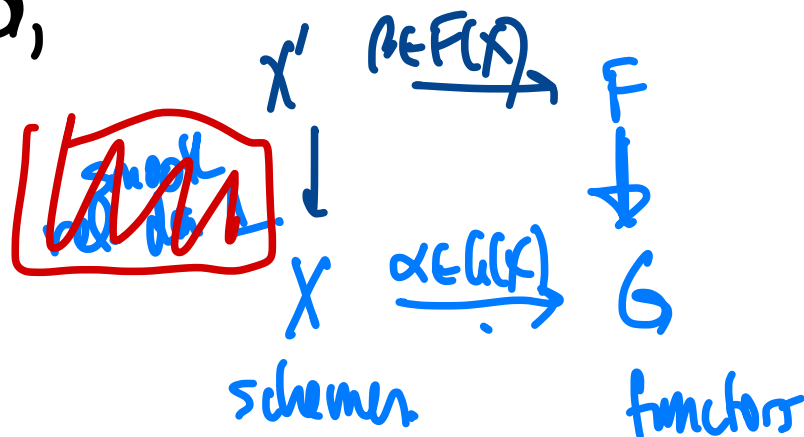
Examples:

- representable morphisms of **FUNCTORS / SHEAVES**
 - open sub**FUNCTORS** (open sub**SHEAVES**)
- An open sub**FUNCTOR** of a **SHEAF** is also a **SHEAF**. Why? (Walking around exercise)

— What is the definition of a locally closed subFUNCTOR?

— What does it mean that a morphism of FUNCTORS $F \rightarrow G$ is ~~smooth of~~ ~~relative dimension 1~~? (Answer: $F \rightarrow G$

is representable, and for any map from a scheme X to G ,



Main Theorem from last day that you can truly (hope to) prove for yourself:

If \mathcal{F} is a **SHEAF** which has a cover by open sub**FUNCTORS** which are representable, then \mathcal{F} itself is representable.

This isn't abstract nonsense — the proof tells you how to construct the representing scheme!

Aside

Interesting Problem from last day, formulated the way I want.

If \mathcal{F} is a **FUNCTOR**, define the intersection of two given open sub**FUNCTORS**.

If \mathcal{F} is a **SHEAF**, define the union of a given arbitrary collection of open sub**SHEAVES**
= open sub**FUNCTORS**.

(Is this the right question?)

Back to reality, down from the clouds ...

Let's use this point of view to
cheaply do more things.

Grassmannian bundles.

Suppose $\begin{matrix} \mathcal{E} \\ | \\ X \end{matrix}$ is a rank n

locally free sheaf, not necessarily trivial.

We should be able to define $\begin{matrix} \mathcal{G}(k, \mathcal{E}) \\ \downarrow \\ X \end{matrix}$

Let's do it!

First step: What is $B \rightarrow \mathcal{G}(k, \mathcal{E})$?



The functor.

$$B \rightarrow \text{Gl}(k, \mathcal{E})$$



$$g: B \rightsquigarrow$$

$$p: B \rightarrow X$$

and a surjection

$$p^* \mathcal{E} \rightarrow V$$

rank k

First:

If $\mathcal{E} \rightarrow \mathcal{O}^{\oplus n}$ is trivial, then g is representable (by $X \times \text{Gl}(k, n)$)

Second (general case): \mathcal{E} is locally trivial;

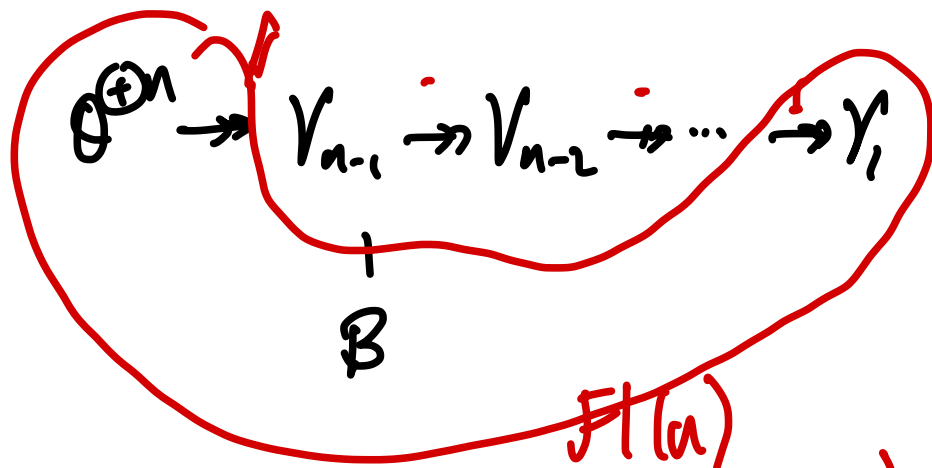
say $\{U_i\}$ is an open cover of X , for which $\mathcal{E}|_{U_i}$ is trivial. Then $g_i = \text{Gl}(k, \mathcal{E}|_{U_i})$ is representable

The G_i cover $G(k, \mathbb{A}^n)$.

Exercise show that the flag variety $FL(n)$ is representable.

The functor:

$$B \longmapsto$$



Flag bundles?

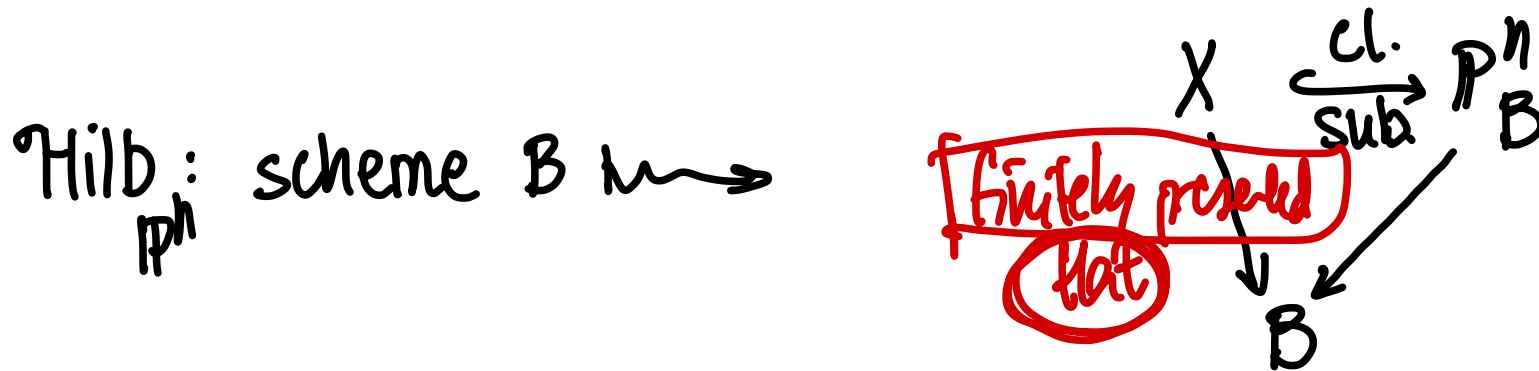
Partial flag bundles?

$\downarrow FL(n-1)$
 \mathbb{P}^{n-1}

$FL(n-1, \ker \mathbb{A}^n \rightarrow \mathbb{A}^1)$
 $G(k, \mathbb{A}^n)$

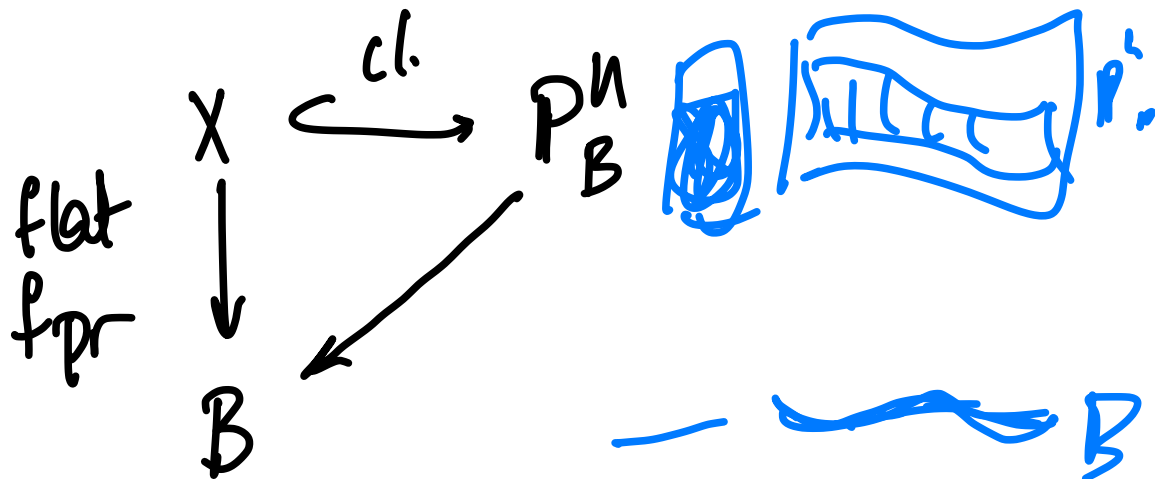
Another example (and the first time we meet the Hilbert scheme)

Define the Hilbert FUNCTOR for \mathbb{P}^n as:



Theorem (another day) $\text{Hilb}_{\mathbb{P}^n}$ is representable,
by a scheme we call the Hilbert scheme
of \mathbb{P}^n , $\text{Hilb } \mathbb{P}^n$.

Given a family



The Hilbert polynomial is a locally constant function of B .

$$B \rightarrow \mathbb{Q}[t]$$

Corollary $\text{Hilb } \mathbb{P}^n = \coprod_{p \in \mathbb{Q}[t]} \mathcal{H}ilb_{p(t)} \mathbb{P}^n$

In fact, we will show that $\mathcal{H}ilb_{p(t)} \mathbb{P}^n$ is (representable by) a projective scheme.

Exercise Suppose $X \rightarrow Y$ is a (Zariski)

\mathbb{P}^n -bundle. Define $\text{Hilb}_{X \rightarrow Y}$ parametrizing

locally closed subschemes of X over Y ; it should be a Hilb \mathbb{P}^n -bundle over Y .

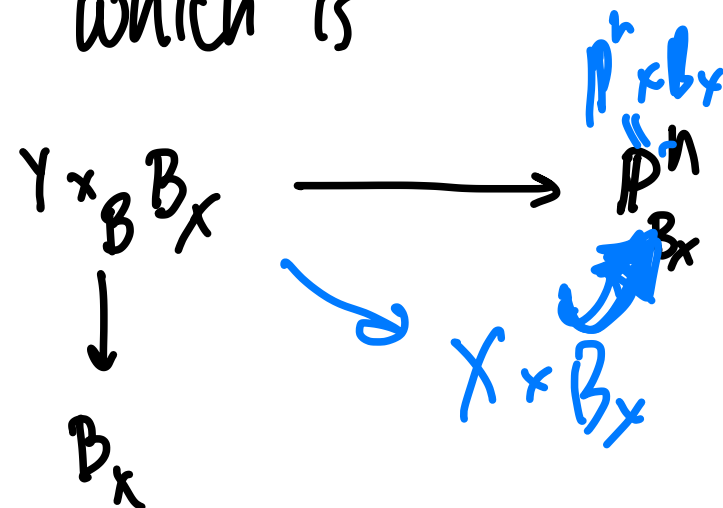
Exercise Suppose $X \xrightarrow{\text{cl.}} \mathbb{P}^n$ is a closed subscheme of \mathbb{P}^n .

Suppose $Y \xrightarrow{\text{cl.}} \mathbb{P}_B^n$ is a flat f.p. family of closed subscheme of \mathbb{P}_B^n .

flat finitely presented family of closed subscheme of \mathbb{P}_B^n .

Define a closed subscheme $B_x \subset B$ which is maximal among those for which

maximal among those for which



factors through $X \times B_x$.

Hilb $X \xrightarrow{\text{loc. cl.}} \text{Hilb } \mathbb{P}^n$
 surjunct.

Possible attack: universal property.

Define a category whose objects are $Z \rightarrow B$ schemes

such that $Z \times_B Y \rightarrow \mathbb{P}^n \times Z$ factors through $X \times Z$.

Show that there is a universal such $Z \rightarrow B$.

Show that this is a closed embedding.

Exercise

If $\text{Hilb } \mathbb{P}^n$ is representable, and $Y \xrightarrow{c} \mathbb{P}^n$,
show that $\text{Hilb } Y$ is representable, by describing
why $\text{Hilb } Y \hookrightarrow \text{Hilb } \mathbb{P}^n$ is a closed embedding.

(Unconditional statement: show that

$\text{Hilb } Y \rightarrow \text{Hilb } \mathbb{P}^n$ is a "closed embedding"

representable subfunctor.)

Exercise If $X \rightarrow Y$ is a locally projective
morphism, define $\text{Hilb}_{X \rightarrow Y}$.

Examples of Hilbert Schemes

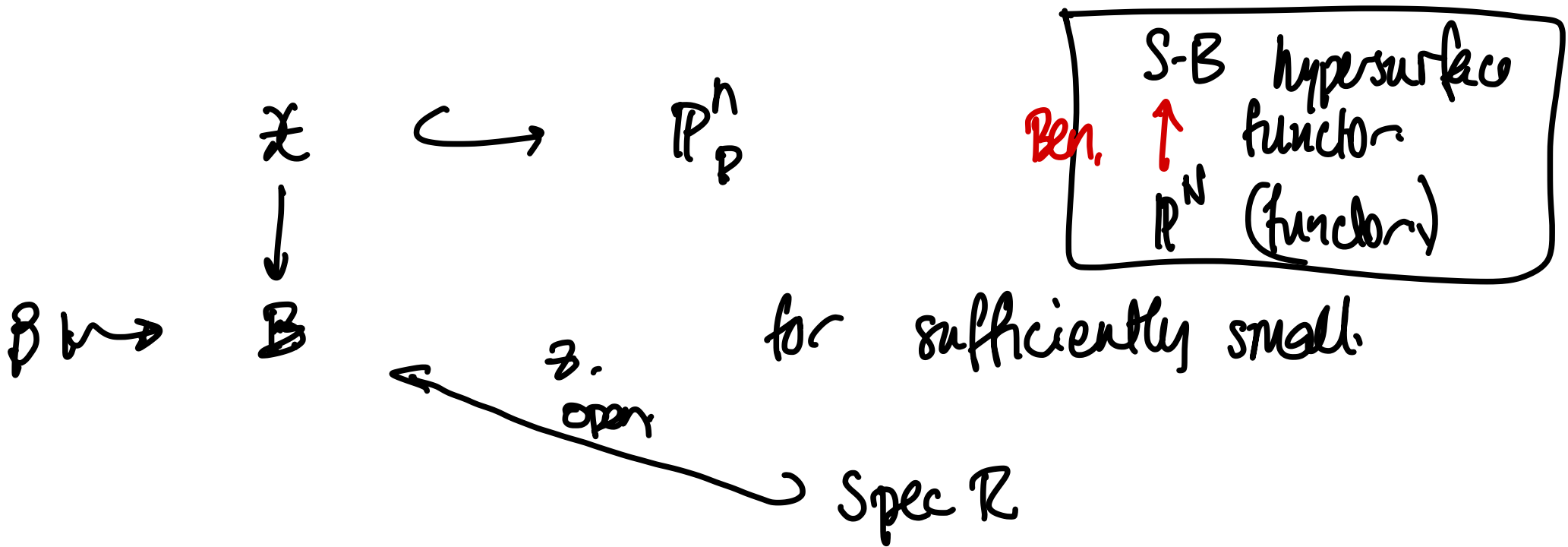
Spencer-Ben
Grassmannian
FUNCTOR

Spencer-Ben
hypersurface
FUNCTOR

$B \rightarrow$

degree d in $\mathbb{P}^n \leftarrow$

$$\begin{aligned} &? x_0^d + \dots + \dots + ? x_n^d \\ &= \mathbb{P}^N \qquad N = \binom{n+d}{d} - 1 \end{aligned}$$



\exists local description:

$$V(a_{x_0} x_0^d + \dots + a_{x_n} x_n^d) \subset \mathbb{P}^n_{\mathbb{R}}$$

$a_i \in \mathbb{R}$; for all points of $\text{Spec } \mathbb{R}$, not all zero.

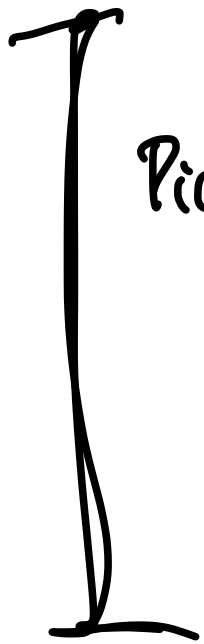
Given

$$B \rightarrow \mathbb{P}^n.$$

want: SB family.

line bundle \mathcal{L} on B
and $N+1$ sections
no common zeros.

$$\mathcal{O}^{\oplus N+1} \rightarrow \mathcal{L}$$



Pick any (\mathcal{L}) locally trivial open cover of B

Get exactly what we want: SB family!

(check: independent of...)

Produce $\mathcal{X} \subset \mathbb{P}^n \times B$

$$\text{On } \mathbb{P}_B^n = B \times \mathbb{P}^n \rightarrow \pi^* \mathcal{O}^{\oplus N+1} \rightarrow \pi^* \mathcal{L}$$

