Moduli Spaces in Algebraic Geometry

Math 245A (winter 2022)

Jan. 10, 2022.

Last day: the Grassmannian exists!
is defined!
is representable!

Today: We abstract some lessons, so we may apply them to other moduli spaces

Early last week: impress your friends
late last week: amaze your friends
early this week: your friends are concerned
late this week: your friends are frightened
end of quarter: you have no friends

Defining the Grassmannian $G(k,n)$ as the thing that represents the $\underbrace{\text{contravariant functor } (\text{Schemes}) \to (\text{Sets})}_{\text{functor}}$, $\mathcal{G}$

FUNCTOR?

$$\mathcal{G}: \quad B \quad \mapsto \quad \begin{array}{c} \mathcal{O}_B^{\oplus n} \twoheadrightarrow V \\ | \\ B \end{array}$$

V rank k locally free

Last day: there is such a thing.

apology for change of letter

What we did: $\mathcal{G}_I$ is a FUNCTOR represented by $\mathbb{A}^{k(n-k)} =: G(k,n)_I$ $\quad$ ($I \subset \{1, \dots, n\}$, $|I| = k$).

$$\mathcal{G}_I : \quad B \longmapsto \begin{array}{c} \mathcal{O}^{\oplus n} \twoheadrightarrow V \\ | \\ B \end{array} \qquad \text{such that} \quad \mathcal{O}^{\oplus I} \xrightarrow{\sim} V$$

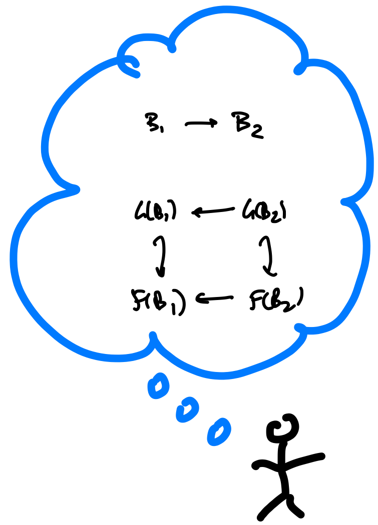
Relationship of $\mathcal{G}_I$ to $\mathcal{G}$ that was relevant?

$\mathcal{G}_I$ is an open subFUNCTOR of $\mathcal{G}$.

Definition $\mathcal{E}$ is a subFUNCTOR of $\mathcal{F}$ if there is implicitly a map (natural transformation) of FUNCTORS

$$i : \mathcal{E} \longrightarrow \mathcal{F}$$

with $\mathcal{E}(B) \xhookrightarrow{i} \mathcal{F}(B)$



($\mathcal{E}(B)$ is naturally a subset of $\mathcal{F}(B)$ for all B)

Definition A subFUNCTOR $G \hookrightarrow F$ is an open subFUNCTOR if for every $\alpha \in F(B)$, there exists a Zariski-open subset $U_\alpha \overset{i}{\subset} B$ such that

For any $Y \xrightarrow{f} B$

such that

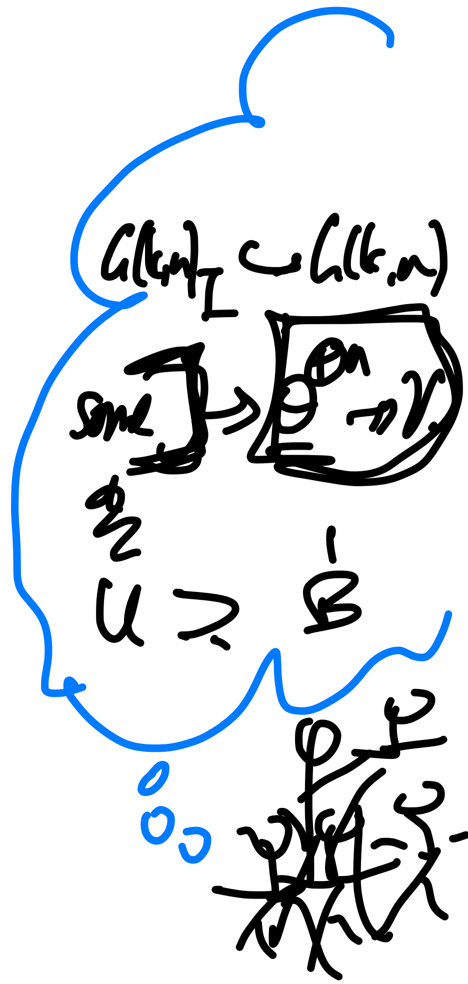
$j^*\alpha \in F(Y)$

$G(Y)$

then
$$\begin{tikzcd} & U \arrow[d, "i"] \\ Y \arrow[ur] \arrow[r, "j"] & B \end{tikzcd}$$

$\in G(U_\alpha)$

$\alpha \in F(B) \longrightarrow i^*\alpha \in F(U_\alpha)$



Remark This looks a bit like an universal property.

~~ABIDE~~

Definition Representable map of FUNCTORS
(map → transformation.)

$$\begin{array}{ccc} F_1 & \longrightarrow & F_2 \\ \vdots ? & & \uparrow \\ ?? & \dashrightarrow & X \text{ scheme} \end{array}$$

$F_1 \to F_2$ is representable if for every X scheme.

$\dashrightarrow$:= map of schemes

?? this is also a scheme.

yoneda: full subcategory.

(Schemes) $\longleftrightarrow$ (FUNCTORS) has fibered product (cheap).

Then $F_1 \to F_2$ is a representable map of functors.

last day.

Example $\mathcal{G}_I$ is an open subFUNCTOR of $\mathcal{G}$.

Claim If $\mathcal{G}$ is a representable FUNCTOR, and $\mathcal{H} \hookrightarrow \mathcal{G}$ is an open subFUNCTOR, then $\mathcal{H}$ is representable.

Pf Say $\mathcal{G}$ is represented by Y.

$\alpha \in \mathcal{G}(Y)$ universal family

Then

$$\begin{tikzcd} \boxed{X} \arrow[r, hook, "\text{open}"'] \arrow[d] & Y \arrow[d] \\ \mathcal{H} \arrow[r, hook, "\text{open}"] & \mathcal{G} \end{tikzcd}$$

$\exists\ \overset{X}{U_\alpha} \subset Y$

Dot dot dot ...

Remark: With the Grassmannian, we went the other way, showing representability of $\mathcal{G}$ using representability of subFUNCTORS $\mathcal{G}_I$ plus more thinking.

Definition The intersection (= fibered product) of two subFUNCTORS (of $\mathcal{F}$), e.g.

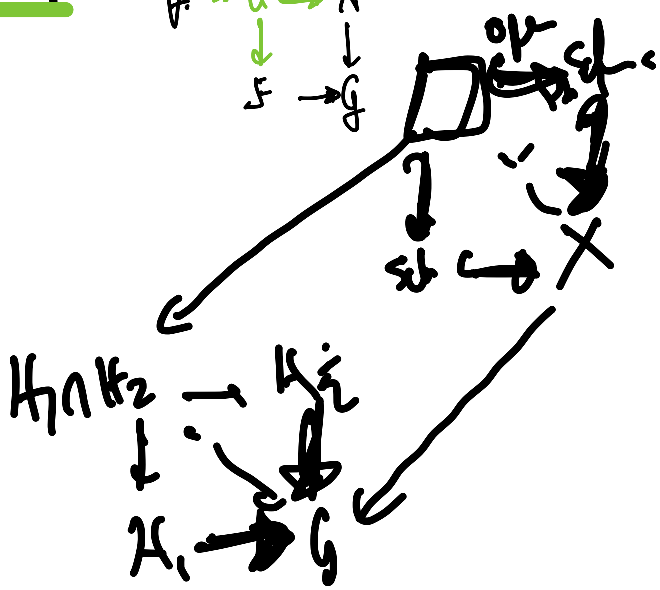
$$\mathcal{F}_I \cap \mathcal{F}_J.$$

$$(\mathcal{F}_I \cap \mathcal{F}_J)(B) = \mathcal{F}_I(B) \cap \mathcal{F}_J(B) \text{ in } \mathcal{F}(B)$$

Claim The intersection of two open subFUNCTORS $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{G}$ (call it $\mathcal{H}_1 \cap \mathcal{H}_2$) is an open subFUNCTOR of $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{G}$.

Proof Spencer's defn of open subfunctor:

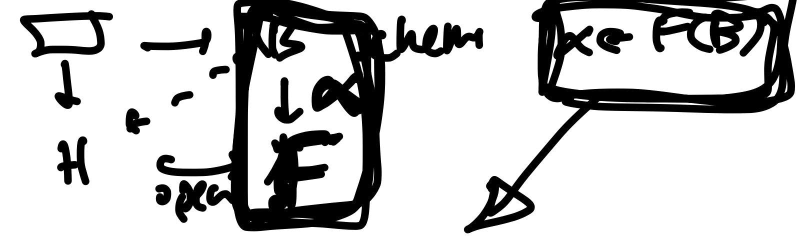
$$\begin{tikzcd} \forall : U \arrow[r, hook, "\text{open}"] \arrow[d] & X \arrow[d] \\ \mathcal{F} \arrow[r] & \mathcal{G} \end{tikzcd}$$



Given:

$\mathcal{H}_1 \to \mathcal{G}$ $\quad$ $\mathcal{H}_2 \to \mathcal{G}$ both rep.

$\alpha : B \to F$

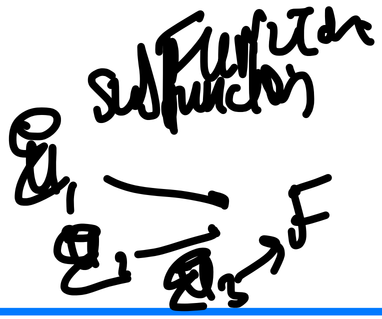


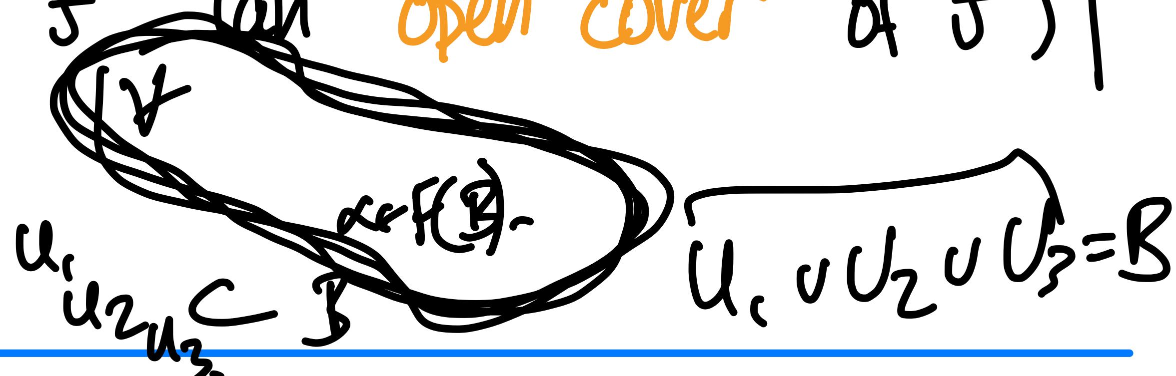
Definitions for you to make:

- The union of two open subFUNCTORs of F.

  Claim: is an open subFUNCTOR of F.

- When a collection of open subFUNCTORs of F covers F (an "open cover" of F)





Remark We have defined something resembling a Zariski topology on the category of FUNCTORS, extending the notion on schemes. (!?)

$\alpha \in F(B)$

$$B \xrightarrow{\alpha} F$$

Define **FUNCTOR** F:

$B \longrightarrow$ Those maps $B \to \mathbb{P}^1$
so that $\exists$ factorization

$\mathbb{A}^1 \amalg \mathbb{A}^1$

$B \to \mathbb{A}^1$ $\qquad\qquad$ $B \to \mathbb{A}^1$

1) F not a sheaf

2)

Back to the Grassmannian...

We showed $\mathcal{G}_I$ was representable.

Hence $\mathcal{G}_I \cap \mathcal{G}_J$ ("$\mathcal{G}_{I \text{ and } J}$") is representable.

The next step:

We showed: $\mathcal{G}_I \cup \mathcal{G}_J$ is representable.

Not true:

~~If $\mathcal{F}, \mathcal{F}'$ are open subFUNCTORS of $\mathcal{F}$, both representable, then so is $\mathcal{F} \cup \mathcal{F}'$.~~

Repeating the argument.

BIG Proposition Assume $\mathcal{F}$ is a FUNCTOR that is a SHEAF.

Suppose $\mathcal{F}'$ and $\mathcal{F}''$ are open subFUNCTORS of $\mathcal{F}$, that are representable by X' and X'', say.
Then $\mathcal{F}' \cup \mathcal{F}''$ is representable.

Proof

$\boxed{\mathcal{F}' \cap \mathcal{F}''}$ represented by $U \xrightarrow{\text{open}} X'$

$U \to X''$

$G_{I \text{ and } J}$

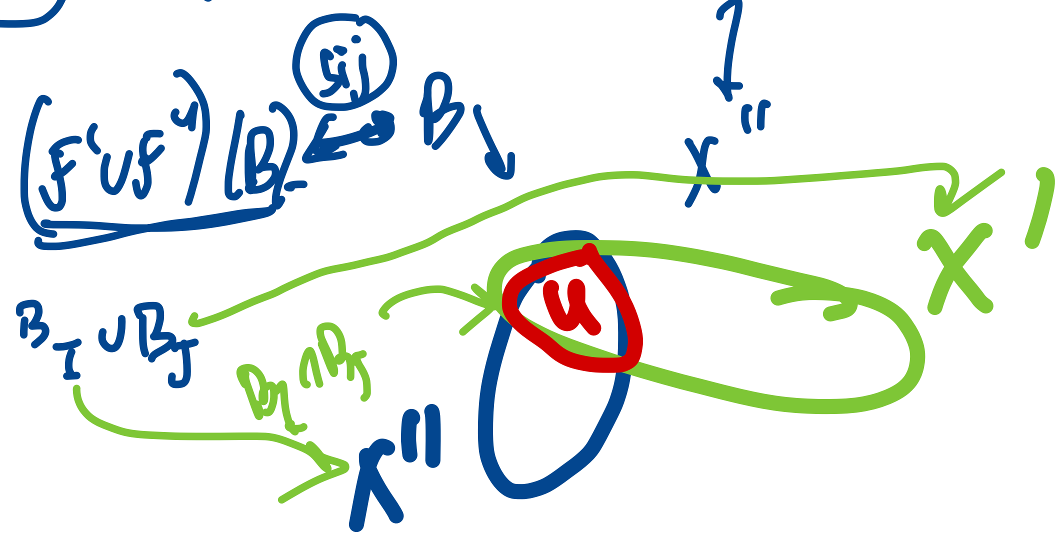
$G_I \qquad G_J$

$G_{I \text{ or } J}$

$\to V$

B

rep?



Definition We say that a FUNCTOR $\mathcal{F}$ is a sheaf (SHEAF?) in the Zariski topology if for all schemes X, the presheaf on X of sets

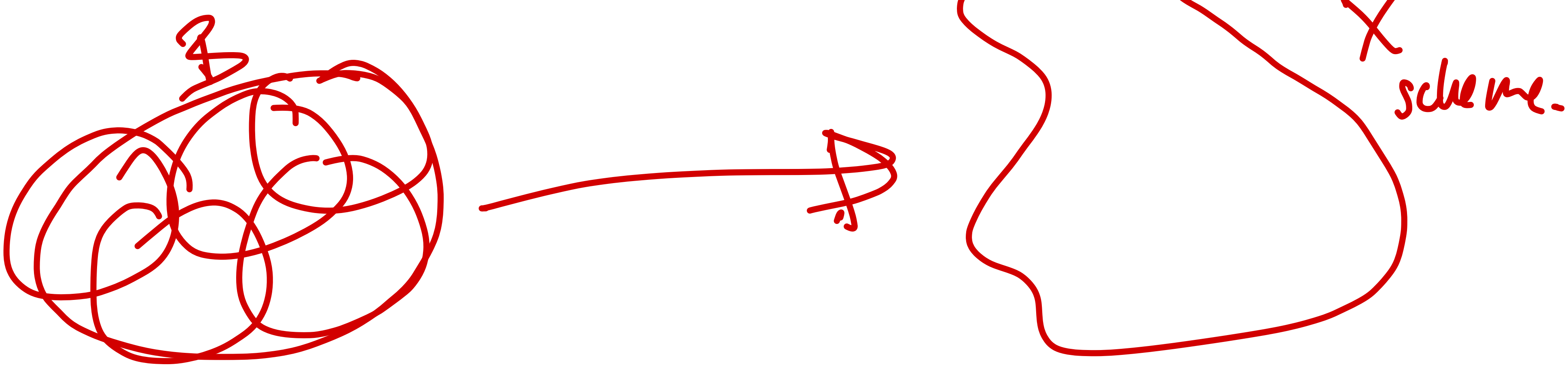
(contravariant functor from the category of open subsets of X to $\mathcal{U}$)

$$\underline{U} \longmapsto \mathcal{F}(\underline{U}) \qquad (U \hookrightarrow X \text{ Zariski open})$$

forms a sheaf

Important Exercise: Representable FUNCTORS are SHEAVES. Translation: maps to schemes glue.

$B = \bigcup B_i$



Theorem If a FUNCTOR $\mathcal{F}$ is a SHEAF that has an open cover by representable subFUNCTORS then $\mathcal{F}$ is a representable FUNCTOR.

Application

Claim: The Grassmannian FUNCTOR is a SHEAF.

Proof:

Corollary: As G_I is representable, the Grassmannian is representable.

Application: Fibered products of schemes are schemes ("new" proof)

Proof: Suppose
$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$
is a diagram of schemes.

We wish to show that the FUNCTOR

$\mathcal{F} : B \longmapsto \mathrm{Maps}(B,X) \times_{\mathrm{Maps}(B,Z)} \mathrm{Maps}(B,Y)$ is representable.

Three worthwhile exercises:

Check: $\mathcal{F}$ is a SHEAF.

Check: $\mathcal{F}$ is covered by $\left\{ \begin{array}{ccc} & & \mathrm{Spec}\, B \\ & & \downarrow \\ \mathrm{Spec}\, A & \to & \mathrm{Spec}\, C \end{array} \right\}$'s.

Check: These are representable by $\mathrm{Spec}\, A \otimes_C B$.