# Moduli Spaces in Algebraic Geometry

Math 245 A (Winter 2022)

Jan. 7, 2022.

## Unintentional falsehood from last time:

I falsely stated that for any ring $R$ and any map

$$R^n \xrightarrow{\phi} R^k$$

$\ker \phi$ is free.

(Thanks to Matthew Hase-Liu for catching this.)

Fun fact: Some time today we will find some $N$ independent of $\phi$ so that there are $f_1, \dots, f_N$ so that $\ker(\phi)_{f_i}$ is a free $R_{f_i}$ module, and $(f_1, \dots, f_N) = 1$.

## Question from last time

contravar. sch. → sets

Consider the space "Pic" given by the functor

$$B \longrightarrow \{\text{line bundles on } B \text{ up to isomorphisms}\}$$

It must be a group scheme; we have

$$\mathrm{Pic} \times \mathrm{Pic} \longrightarrow \mathrm{Pic}.$$

What is it?

No such scheme!

⟶ Group functor.

Goal today:

$$\boxed{\text{the Grassmannian exists!}}$$

Translation:

controvariant (Schemes) → (Sets)

The Grassmannian functor is representable!

<u>Which</u> Grassmannian functor?

Reminder:

$/\mathbb{Z}$

$/F$

Fix $k, n$.

$$\mathcal{F} : (\text{Schemes}) \xrightarrow{\text{contravariant}} (\text{Sets})$$

$$B \rightsquigarrow \left\{ \begin{array}{c} 0 \to \mathcal{N} \to \mathcal{O}^{\oplus n} \twoheadrightarrow \mathcal{V} \to 0 \\ \Big| \quad\quad \text{rank } k \\ B \end{array} \right\}$$

($\mathcal{V}$, along with $n$ sections of $\mathcal{V}$, which "generate it")

$$\begin{array}{ccc} \{\mathcal{O}^{\oplus n} & \twoheadrightarrow & \mathcal{V}\} \\ \| & & \wr ? \\ = \{\mathcal{O}^{\oplus n} & \twoheadrightarrow & \mathcal{V}'\} \end{array}$$

means ...

**Example:** $k=1$. $\qquad \mathcal{O}^{\oplus n} \to \mathcal{L}$

$G(1,n) =$ line bundle, $n$ sections with no common zero

$= \mathbb{P}^{n-1}$

(**Exercise:** think this through)

**Example:** $G(k,n) \xrightarrow{\sim} G(n-k,n)$

$$0 \to \underset{\text{rank } n-k}{\mathcal{W}} \to \mathcal{O}^{\oplus n} \to \underset{k-}{\mathcal{V}} \to 0$$

dualize it $\qquad \mathcal{O}^{\oplus n} \to \mathcal{W}^{\vee}$

**Example:**

$$\mathcal{O}^{\oplus n} \twoheadrightarrow V \quad \text{rank } k \qquad\qquad \mathcal{O}^{\oplus \binom{n}{3}} \twoheadrightarrow \Lambda^3 V$$

$$G(k,n) \to G\left(\binom{k}{k}, \binom{n}{k}\right)$$

$$G(k,n) \to G\left(1, \binom{n}{k}\right) = \boxed{\mathbb{P}^{\binom{n}{k}-1}}$$

Plücker morphism

(soon: closed embedding)

Back to the construction...

For $I \subset \{1, \dots, n\}$, $|I| = k$

Consider the (moduli) functor

$$F_I : B \rightsquigarrow \left\{ \begin{array}{c} \mathcal{O}^{\oplus n} \twoheadrightarrow V \text{ rank } k \\ \uparrow \quad \nearrow_{\sim} \\ \mathcal{O}^{\oplus I} \end{array} \right. \text{ such that}$$

<u>Claim</u>

Then $F_I$ is representable by $\mathbb{A}^{k(n-k)}$. Call it $G(k,n)_I$.

Here's why:

$$F_I : B \rightsquigarrow \left\{ \mathcal{O}^{\oplus(n-k)} \xrightarrow[n-k]{} \mathcal{O}^{\oplus k} \right\} \dots$$

$k \times (n-k)$ functions

//

Another translation of $F_I$ $\quad (I = \{i_1, \ldots, i_k\}$ $\subset \{1, \ldots, n\}$.)

$$B \rightsquigarrow \quad \mathcal{O}^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} V$$

with the section

$s_{i_1} \wedge s_{i_2} \wedge \cdots \wedge s_{i_k}$ of $\Lambda^k V = \det V$

line bundle

vanishing nowhere.

That's where we left off on Wednesday.

Define

$$\mathcal{F}_{I \text{ and } J} : B \longmapsto \left\{ \begin{array}{c} (s_1,\dots,s_n) \\ \mathcal{O}^{\oplus n} \twoheadrightarrow V \\ \downarrow \qquad \text{rank } k \\ B \end{array} \right\}$$

$I = \{i_1,\dots,i_k\}$ $J = \{j_1,\dots,j_k\}$

where the sections $s_{i_1} \wedge \cdots \wedge s_{i_k}$ and $s_{j_1} \wedge \cdots \wedge s_{j_k}$ of $\det V$ **both** vanish nowhere.

**Claim:** $\mathcal{F}_{I \text{ and } J}$ is representable, by $G(k,n)_{I \text{ and } J} \overset{\text{open}}{\hookrightarrow} G(k,n)_I \cong \mathbb{A}^{k(n-k)}$

$G(k,n)_{I \text{ and } J} \hookrightarrow G(k,n)_J$

Why?

(cont'd next page)

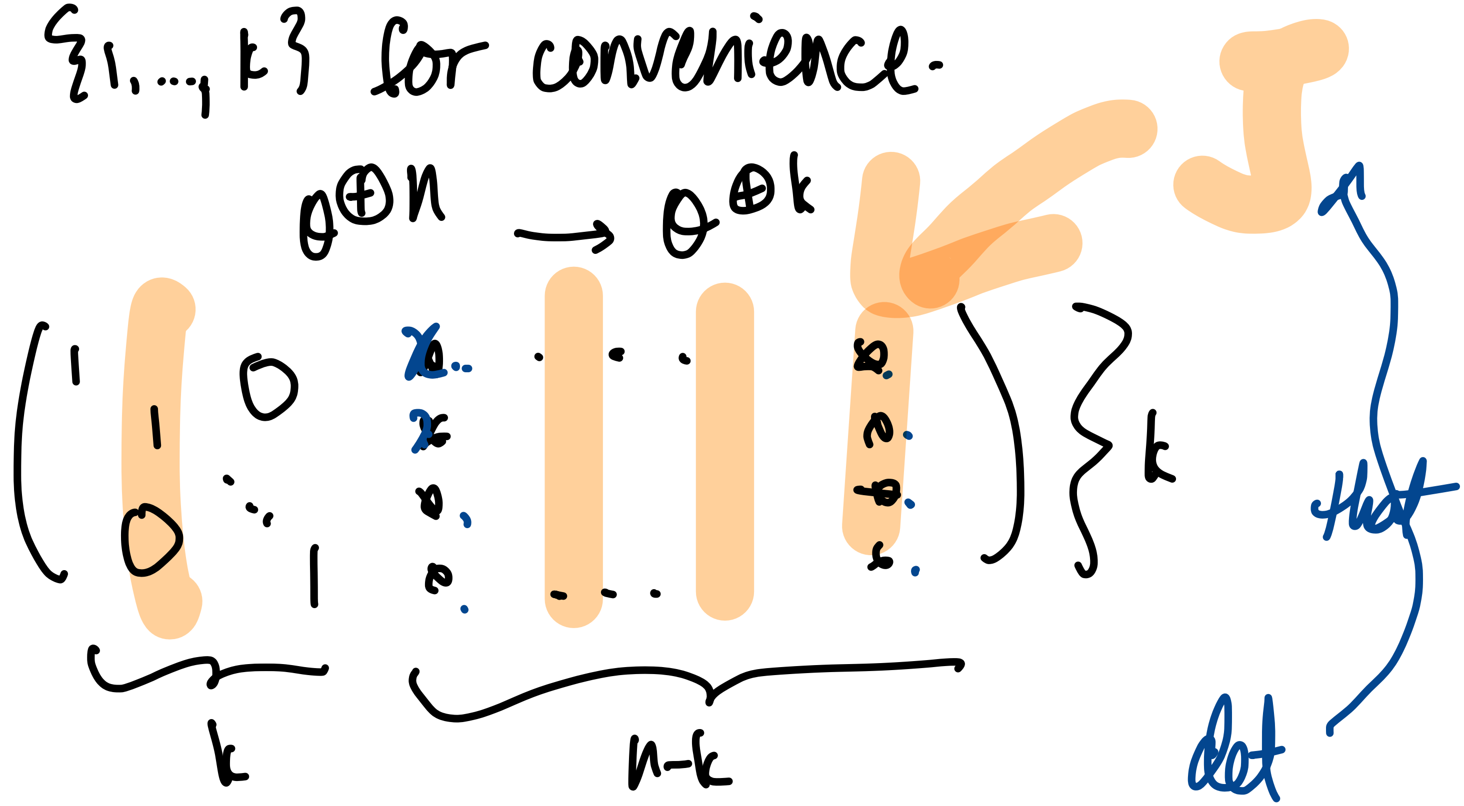
<u>Claim</u>: $\mathcal{F}_{I \cap J}$ is representable, by $G(k,n)_{I \text{ and } J} \overset{\text{open}}{\hookrightarrow} G(k,n)_I$

$\hookrightarrow G(k,n)_J$

Why?

Pf Say $I = \{1, \dots, k\}$ for convenience.

$A^{k(n-k)}$ $\qquad \mathcal{O}^{\oplus n} \to \mathcal{O}^{\oplus k}$



$$\left( \underbrace{\begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix}}_{k} \ \underbrace{\begin{matrix}x & \cdots & x\\ x & & x\\ x & & x\\ x & \cdots & x\end{matrix}}_{n-k} \right) \Big\} k$$

J

that

det

Which Open subset?

$\boxed{s_{j_1} \wedge \cdots \wedge s_{j_k} \text{ nonzero.}}$

$\{j_1, \dots, j_k\} \subset \{1, \dots, n\}$.

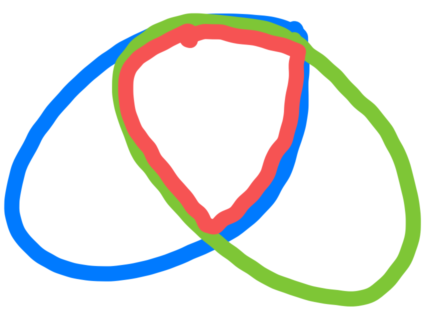
Define $\mathcal{F}_{I \text{ or } J}$ :

$$B \rightsquigarrow \mathcal{O}^{\oplus n} \xrightarrow{(s_1,\dots,s_n)} \underset{\text{rank } k}{V}$$

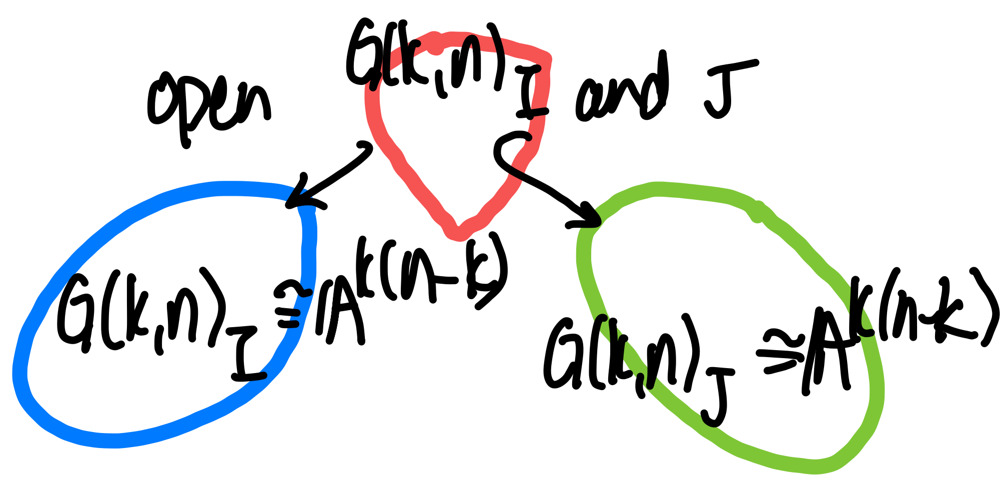
where at every point $p$ of $B$, $\overbrace{s_{i_1}\wedge\cdots\wedge s_{i_k}}^{\text{section of } \det V}(p) \neq 0$

or $s_{j_1}\wedge\cdots\wedge s_{j_k}(p) \neq 0$.

Then $\mathcal{F}_{I \text{ or } J}$ is representable by $G(k,n)_{I \text{ or } J}$ :=

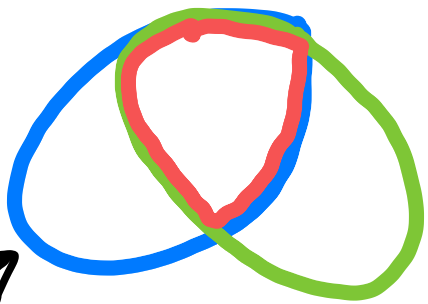


built from

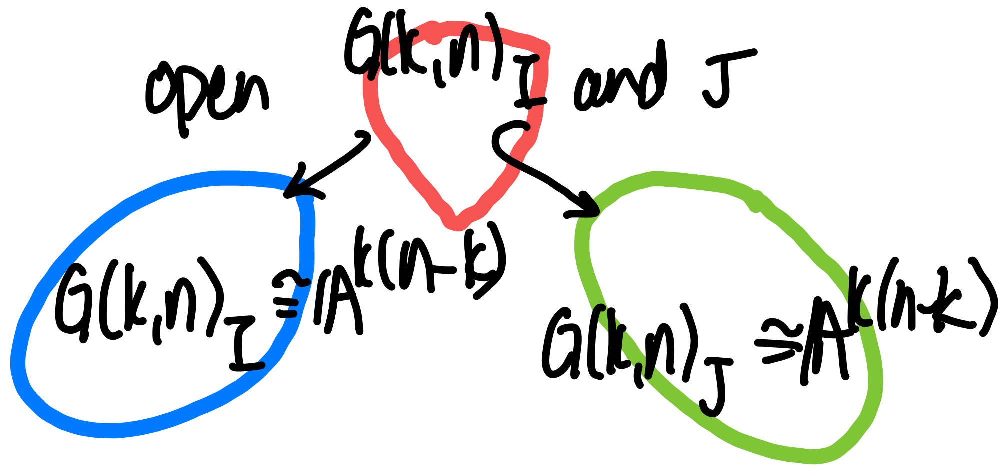


## Claim

$\mathcal{F}_{I \text{ or } J}$ is representable by $G(k,n)_{I \text{ or } J} :=$



built from




## Proof

Given

$$\begin{array}{c} \Theta^{\oplus n} \to V \\ | \\ B \end{array} \quad \text{in } \mathcal{F}_{I \text{ or } J}$$

$\Rightarrow S_{j_1} \wedge \dots \wedge S_{j_k} \neq 0$

$\Rightarrow (S_{i_1} \wedge \dots S_{i_k} \neq 0)$

$B = B_I \cup B_J$

how to build $B$?

Given $B$ how to build

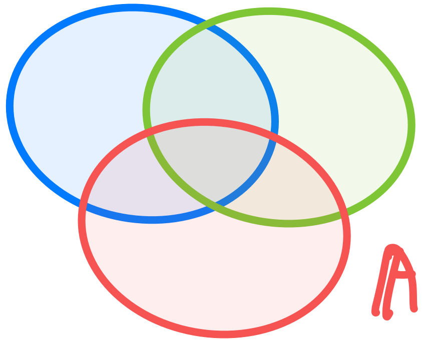
$$\begin{array}{c} \Theta^{\oplus n} \to V \\ | \\ B \end{array}$$

$\boxed{B_{I \text{ and } J}}$

in $\mathcal{F}_{I \text{ or } J}$ ?

**Exercise** $\mathcal{F}_{I \text{ or } J \text{ or } K}$ is representable by:

$\mathcal{O}^{\oplus n} \to V$.



$\mathbb{A}^{k(n-k)}$ $\mathbb{A}^{k(n-k)}$ $\mathbb{A}^{k(n-k)}$

**Exercise** $\mathcal{F}_{\{1,\dots,k\} \text{ or } \dots \text{ or } \{n-k+1,\dots,n\}}$ is representable

$\binom{n}{k}$

$\mathcal{O}^{\oplus n} \twoheadrightarrow V$ rank $k$

$|$

$B$

by something with an open cover by $\binom{n}{k}$ copies of $\mathbb{A}^{k(n-k)}$.

But this is $G(n,k)$: if $\mathcal{O}_B^{\oplus n} \twoheadrightarrow \underset{\text{rank } k}{V}$ then $\Lambda^k(\mathcal{O}^{\oplus n}) \twoheadrightarrow \det V = \Lambda^k V$

## Conclusion / field $F$

$G(k,n)$ is representable by something smooth of dimension $k(n-k)$, with an open cover by $G(k,n)_I \cong \mathbb{A}^{k(n-k)}$

It is an irreducible smooth variety.

$G(k,n)_I$ is the locus where $s_{i_1} \wedge \cdots \wedge s_{i_k} \neq 0$ (as a section of $\det V$). (over $\mathbb{Z}$ ...)

$\hookrightarrow$ universal $\gamma$.

$\mathcal{O}^{\oplus n} \twoheadrightarrow V$ rank .

The Plücker morphism:

$$\begin{tikzcd} G(k,n) \arrow[r, "{[s_1\wedge\cdots\wedge s_k,\ \ldots,\ s_{n-k+1}\wedge\cdots\wedge s_n]}"] & \boxed{\mathbb{P}^{\binom{n}{k}-1}} = G(1, \binom{n}{k}) \end{tikzcd}$$

We know the preimage of each of the "standard open subsets" of $\mathbb{P}^{\binom{n}{k}-1}$:

$$\begin{tikzcd} G(k,n) \arrow[r] & \mathbb{P}^{\binom{n}{k}-1} \\ \boxed{G(k,n)_I} \arrow[u] \arrow[r] & \mathbb{A}^{\binom{n}{k}-1} \arrow[u, "\text{open}"'] \end{tikzcd}$$

$\mathbb{A}^{k(n-k)}$

$I^c$ open set

Final goal: This is a closed embedding
= closed immersion

Corollary: Plucker ~~morphism~~ embedding.

The Grassmannian is projective!

$$G(k,n)_{\{1,\dots,k\}} = \mathcal{I} \longrightarrow \mathbb{A}^{\binom{n}{k}-1}$$

$$= \mathbb{A}^{k(n-k)}$$

$$\begin{pmatrix} 1 & & & * & \cdots & * \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & * & \cdots & * \end{pmatrix}$$

coordinates

$$F\left[\ \ ,\ \dots\ ,\ \ \right] \longrightarrow F[\dots,\dots]$$

$$\frac{S_{i_1}\wedge\cdots\wedge S_{i_k}}{S_1\wedge\cdots\wedge S_k}$$

Why is the map of algebras surjective?

Conclusion: We have constructed the Grassmannian!

(This is our definition)

**Question:** how about the other candidates for the "Grassmannian functor"?

**Spencer/Ben:** $B \mapsto$

$$\begin{tikzcd} X \arrow[r, hook, "\text{cl. sub.}"] & \mathbb{P}^{n-1} \times B \arrow[d] \\ & B \end{tikzcd}$$

such that sufficiently locally on $B$, $\operatorname{Spec} R \subset B$

$X$ is cut out by $n-k$ linear equations

$$r_{11} x_1 + \cdots + r_{1n} x_n = 0$$
$$\vdots$$
$$r_{k1} x_1 + \cdots + r_{kn} x_n = 0$$

That are linearly independant at all points of ~~$B$~~ $\operatorname{Spec} R$.

**Me:** $X \overset{\text{cl.}}{\underset{\text{sub}}{\hookrightarrow}} \mathbb{P}^{n-1}$ over $B$

$X \to B$ flat

finitely presented

fibers all "linear $\mathbb{P}^{k-1}$'s". (yuck!)

**Geometric me:** geometric ...

1) We have:

$\text{me} \overset{=}{\longleftrightarrow} \text{geometric me}$

2) We have: Spencer/Ben

3) We have $G(k,n) \longrightarrow$ Spencer/Ben

$B \rightsquigarrow$ $\boxed{\underset{B}{\mathcal{O}^{\oplus n}} \twoheadrightarrow V}$

$B$

$$\boxed{\mathrm{Sym}^\bullet \mathcal{O}^{\oplus n} \twoheadrightarrow \mathrm{Sym}^\bullet V}$$

graded

$$\underline{\mathrm{Proj}}\ \mathrm{Sym}^\bullet \mathcal{O}^{\oplus n} \xleftarrow[\text{emb}]{\text{closed}} \underline{\mathrm{Proj}}\ \mathrm{Sym}^\bullet V$$

$\mathbb{P}^{n-1}$ trivial

$\mathbb{P}^{k-1}$ more cplxt.