Moduli Spaces in Algebraic Geometry

Math 245 A (winter 2022)

Jan. 5, 2022.

The future of this course:

I may be able to keep the course on zoom while also satisfying Stanford's rules, and meeting in a classroom.

Office hours:

Some on zoom (but when?)

Some in person (but when and where?)

References:

FGA Explained

Mumford's Curves on an Algebraic Surface

Nakajima, Huybrechts, Harris-Morrison, Mukai, ...

Online:

My notes "The Rising Sea"

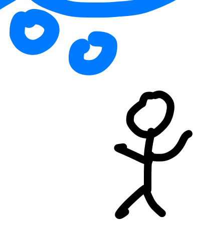
Jarod Alper's notes on moduli spaces and stacks

Where we are:

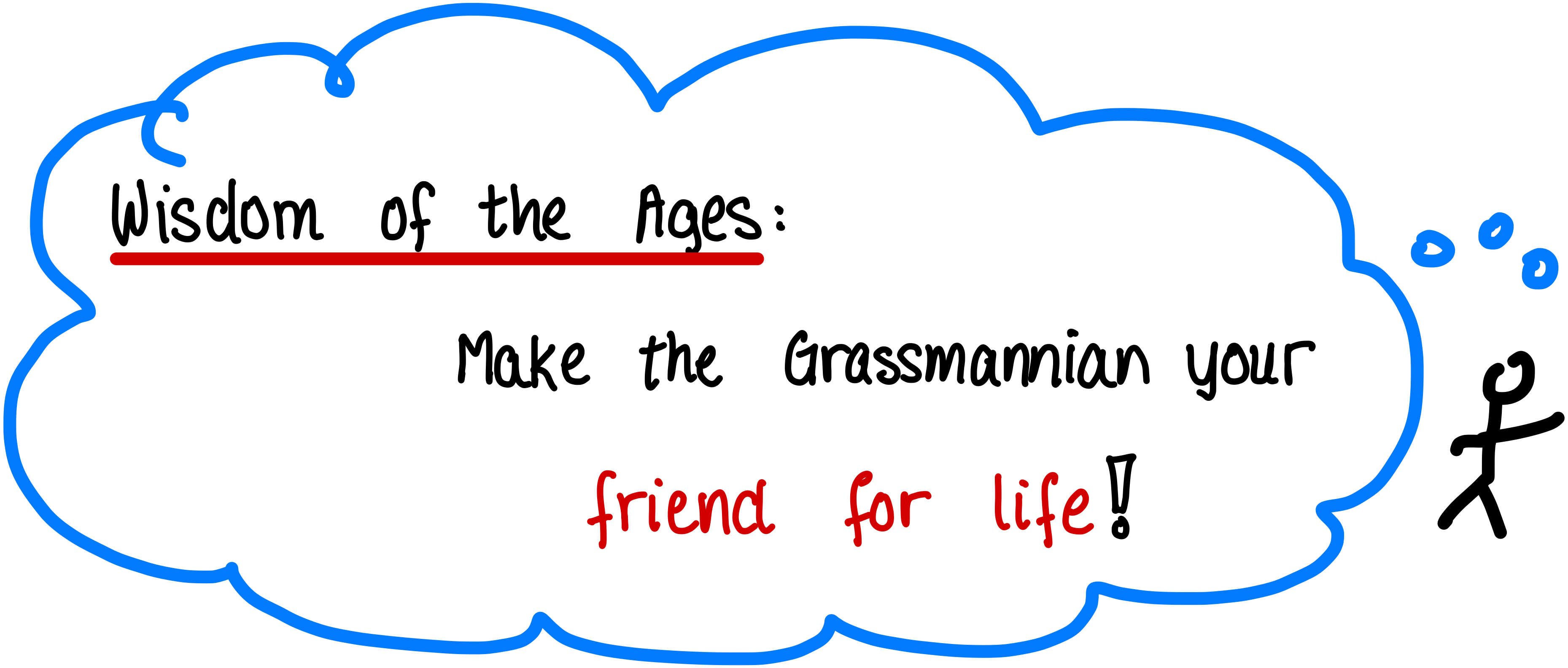
For the sake of concreteness, we will work in the category (Schemes) of schemes.

We can also work in many other categories without change.

If you've seen all this before, you might try to make all this work in the complex analytic category.



Seemingly tiny point from last day (indeed, from linear algebra)



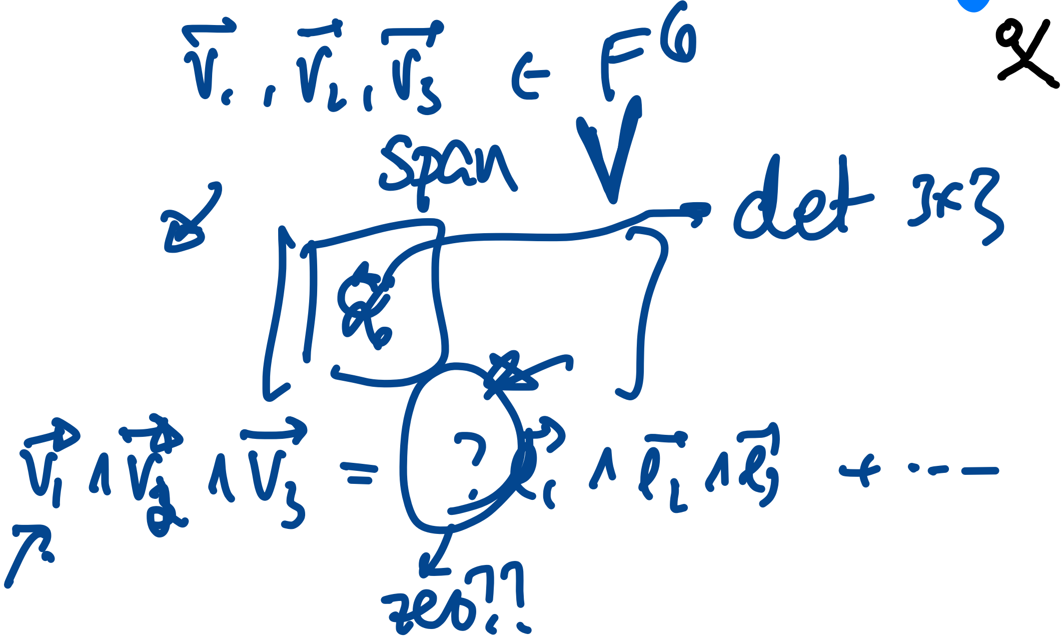
Wisdom of the Ages:

Make the Grassmannian your friend for life!

To classify a k-dimensional sub(vector)space V of F^n (F a field), choose a basis, and put it in reduced echelon form.

$$\underbrace{\begin{matrix} 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{matrix}}_{n=6} \qquad k=3$$

Schubert cells



$\vec v_1, \vec v_2, \vec v_3 \in F^6$

$\vec v_1 \wedge \vec v_2 \wedge \vec v_3 = \; ? \; \vec e_1 \wedge \vec e_2 \wedge \vec e_3 + \cdots$

"Most" are of the form

$$\begin{matrix} \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}} & \begin{matrix} * & * & * \\ * & * & * \\ * & * & * \end{matrix} \end{matrix} \qquad \leftarrow F^9 \qquad \mathbb{A}^9_F$$

Q: How can you tell?

Back to the program...

We want to construct / study / show the existence of a moduli space of something.

Think:

curves, line bundles

vector bundles, ...

abelian varieties (with polarization)

(semistable) sheaves (on a K3) (with fixed Chern character)

stable map.

complexes of sheaves

More precisely:

A moduli functor $\longrightarrow$ $(\text{Schemes}) \to (\text{Sets})$

contravariant

$B \rightsquigarrow F(B)$ "families of our objects over B"

$B_1 \to B_2 \rightsquigarrow F(B_2) \to F(B_1)$

how we 'pull back' families over B_2 to get families over B_1

Typically: translates to:

moduli of stable (certain nodal)) genus g curves.

$g=3$ projective

geometric fibers are

$$\begin{tikzcd}
C \arrow[d, "\text{flat!!}"', "\text{morphism}"] & \\
B & \arrow[l] \cdot \operatorname{Spec} \bar{k}
\end{tikzcd}$$

moduli of closed subschemes of $\mathbb{P}^n$

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "\text{flat.}"'] & \mathbb{P}^n \times B \\
\underline{B} &
\end{tikzcd}$$

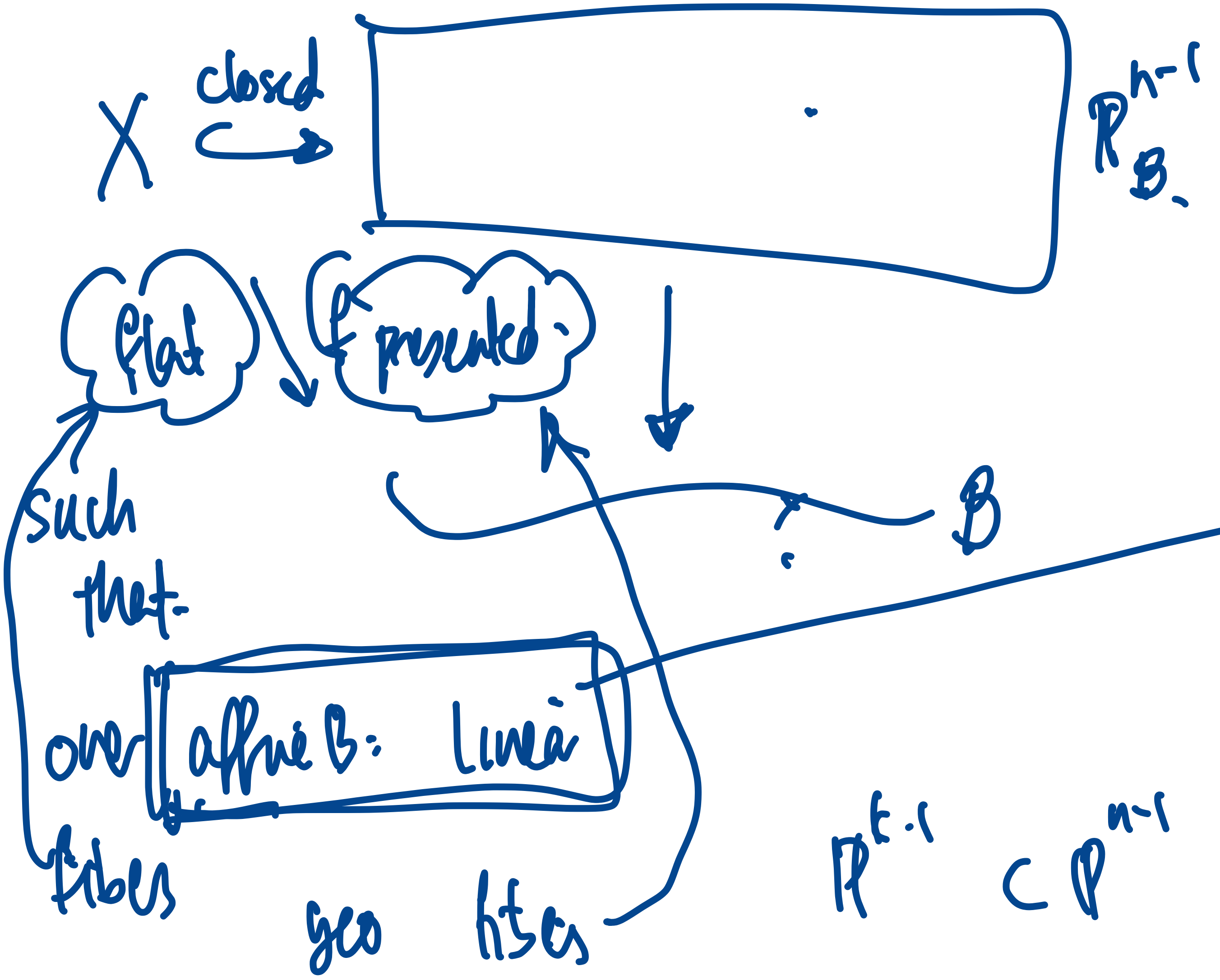
finitely presented.

Let's try this with the Grassmannian $G(k,n)$ or $G(k-1,n-1)$.

"$\mathbb{P}^{k-1}$" $\hookrightarrow \mathbb{P}^{n-1}$

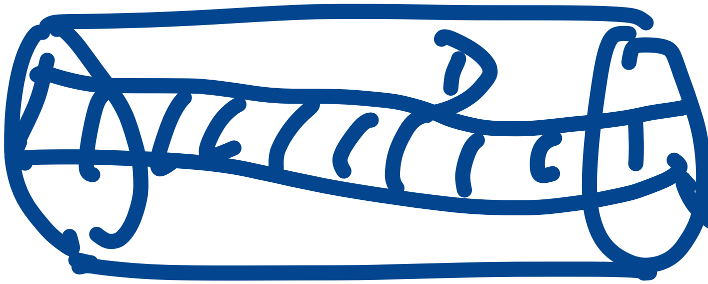
$B \rightsquigarrow$ what?

scheme



Not quite right.

If $B = \operatorname{Spec} A$, $\operatorname{Cl} A \neq 1$.



$\mathbb{P}^n$

degree e.

Tor L

$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$

B

$0 \to \mathcal{O}(e-D) \to \mathcal{O}(e) \to \mathcal{O}_D(e) \to 0$

$0 \to \square \to \mathcal{O}^{\oplus n} \to \mathcal{O} \to 0$

V rank n-1

The moduli functor $\mathcal{F}$ is representable by $\mathcal{M}$

means:

$$\mathcal{F} \overset{\text{iso}}{\longleftrightarrow} \mathrm{Mor}(\cdot\,, \mathcal{M}).$$

$$\begin{tikzcd}
\text{family} \arrow[d, "\mathcal{F}(B)"'] & & & \\
B & \rightsquigarrow & B \arrow[r] & \mathcal{M}
\end{tikzcd}$$

Yoneda.

We say $\mathcal{M}$ is the moduli space for $\mathcal{F}$.

Construction/Existence/Representability of some moduli spaces

Example: moduli space of functions

$$B \mapsto \mathcal{O}(B)$$

$B_1 \to B_2$

$\mathcal{O}(B_1) \leftarrow \mathcal{O}(B_2)$

We know how they pull back.

Is there a space $\mathcal{M}$ so that "functions on B" correspond precisely to maps $B \to \mathcal{M}$?

$\boxed{x}$

Remark: functions form a <u>ring</u>.

The moduli space this forms a ring scheme.

$B \to M = \mathbb{A}^1$

$= \operatorname{Spec} \mathbb{Z}[x]$

Example: Invertible functions:

$$B \rightsquigarrow \mathcal{O}(B)^{\times}$$

$B_1 \to B_2$

(functions vanishing nowhere).

Invertible functions form a group.

The moduli space $\mathcal{M}$ is a group scheme.

$x \in \mathcal{O}(\mathcal{M})$

x^{-1}

e.g., What is the inverse map $\mathcal{M} \to \mathcal{M}$?

$(\pi_1^* x)(\pi_2^* y)$

What is the product $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$?

$\mathcal{M} \xleftarrow{\pi_1} \mathcal{M}\times\mathcal{M}$, $\mathcal{M} \xleftarrow{\pi_2} \mathcal{M}\times\mathcal{M}$

Why is the product commutative? Associative?

(What is $\mathcal{M}$?)

Example: The group of line bundles.

$B \longmapsto \mathrm{Pic}\, B.$ (line bundles on B up to isomorphism)

pullbacts are okay.

Line bundles form a group. Hence:

The moduli space (call it $\underline{\mathrm{Pic}}$) is thus a group scheme.

The Grassmannian Exists

The moduli functor: n, k fixed.

$\mathcal{F}$ is the functor $B \rightsquigarrow$

rank n-k loc. free sheaf ~~vector bundle~~ — rank k ~~vector bundle~~ loc. free sheaf

$$\begin{tikzcd}
\left\{ 0 \arrow[r] & \mathcal{W} \arrow[r] & \mathcal{O}^{\oplus n} \arrow[r] \arrow[d, dash] & \mathcal{V} \arrow[r] & 0 \right\} \big/ \text{isomorphism.} \\
& \text{on} & B & &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{W} \arrow[r] \arrow[d, "\sim"'] & \mathcal{O}^{\oplus n} \arrow[r] \arrow[d, equal] & \mathcal{V} \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \mathcal{W}' \arrow[r] & \mathcal{O}^{\oplus n} \arrow[r] & \mathcal{V}' \arrow[r] & 0.
\end{tikzcd}$$

same.

You say: That's not what I was expecting!

Grothendieck replies:

thm

$$\mathcal{O}^{\oplus n} \xrightarrow{\phi} V$$

rank k.

Then. kernel

$$0 \to \ker\phi \to \mathcal{O}^{\oplus n} \to V \to 0$$

loc free.

$$R^n \to R^k \to 0.$$

kernel is iso to R^{n-k}.

easy!!

$$0 \to W \xrightarrow{\alpha} \mathcal{O}^{\oplus n}$$

loc free rank n-k.

$$R^{n-k} \hookrightarrow R^n$$

coker α not nec. loc <u>free</u>!!

$$0 \to R \longrightarrow R$$

$$0 \to k[t] \xrightarrow{\times t} k[t] \to k[t]/(t) \to 0$$

$$0 \to \Theta(-D) \to \Theta \to \Theta_y \to 0$$

That's too hard for me. Instead:

Define $\mathcal{F}_{\{1,\dots,k\}}$

n sections of $\mathcal{V}$ $\quad s_1,\dots,s_n \in \mathcal{V}(B)$

$$\begin{tikzcd}
\mathcal{O}^{\oplus n} \arrow[r] \arrow[d, dash] & \mathcal{V} \arrow[r] & 0 \\
B & &
\end{tikzcd}$$

where "the first k summands" map isomorphically onto $\mathcal{V}$:

i.e.,

$$\begin{tikzcd}
\mathcal{O}^{\oplus k} \arrow[d] \arrow[dr, "\sim"] & & \\
\mathcal{O}^{\oplus n} \arrow[r] & \mathcal{V} \arrow[r] & 0
\end{tikzcd}$$

Exercise: This is equivalent to: the section $s_1 \wedge s_2 \wedge \dots \wedge s_k$ of $\det \mathcal{V}$ vanishes nowhere.

Here again is our moduli functor $\mathcal{F}_{\{1,\dots,k\}}$

$$\begin{tikzcd}
\mathcal{O}^{\oplus k} \arrow[d] \arrow[dr, "\sim"] & & \\
\mathcal{O}^{\oplus n} \arrow[r] & V \arrow[r] & 0
\end{tikzcd}$$

Defn.

$$\mathcal{O}^{\oplus(n-k)} \to \boxed{\mathcal{O}^{\oplus k}}$$

$k(n-k)$ functions

What is this moduli space?

$$\begin{pmatrix} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \end{pmatrix}$$

$k(n-k).$

$\mathcal{F}_{\{1,\dots,k\}}$ is representable by $G(k,n)_{\{1,\dots,n\}} :\equiv \cancel{?}\ \mathbb{A}^{k(n-k)}$

MAGIC!

happens somewhere

If $I \subset \{1, \dots, n\}$ has size k, let $\mathcal{F}_I$ be the functor

$B \rightsquigarrow$

$$\begin{tikzcd}
\mathcal{O}^{\oplus n} \arrow[r] \arrow[d, dash] & \mathcal{V} \arrow[r] & 0 \\
B & &
\end{tikzcd}$$

where "$\mathcal{O}^{\oplus I}$" $\longrightarrow \mathcal{V}$ is an isomorphism.

$\mathcal{F}_I$ is representable (by $\mathbb{A}^{k(n-k)}$)

Suppose $I_1, I_2 \subset \{1, \dots, n\}$ both size k

Define the moduli functor

$\mathcal{F}_{I_1 \text{ and } I_2}$:

$$\begin{tikzcd}
\mathcal{O}^{\oplus n} \arrow[r] \arrow[d, dash] & V \arrow[r] & 0 \\
B & &
\end{tikzcd}$$

such that $\mathcal{O}^{\oplus I_1} \xrightarrow{\sim} V$ and $\mathcal{O}^{\oplus I_2} \xrightarrow{\sim} V$.

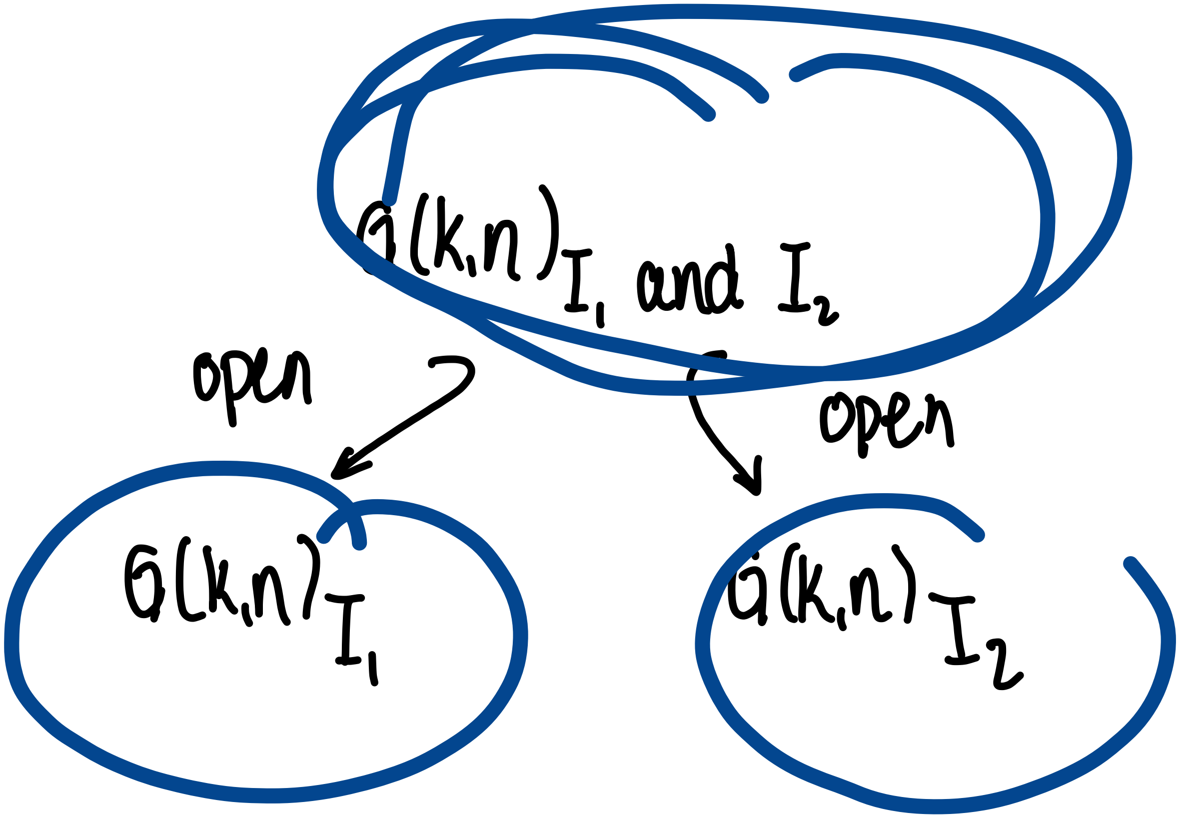
This is representable. By what?

$I_1 = \{1 .. k\}$

$$\begin{pmatrix} 1 & & & & ? & ? & \cdots & ? \\ & 1 & & & ? & ? & \cdots & ? \\ & & 1 & & ? & ? & \cdots & ? \\ & & & 1 & ? & ? & \cdots & ? \end{pmatrix}$$

$U \subset \mathbb{A}^{k(n-k)}$

where $\det(\ ) \neq 0$


$G(k,n)_{I_1}$ and I_2
open
open
$G(k,n)_{I_1}$
$G(k,n)_{I_2}$

Now consider the functor

$\mathcal{G}(k,n)_{I_1 \text{ or } I_2}$

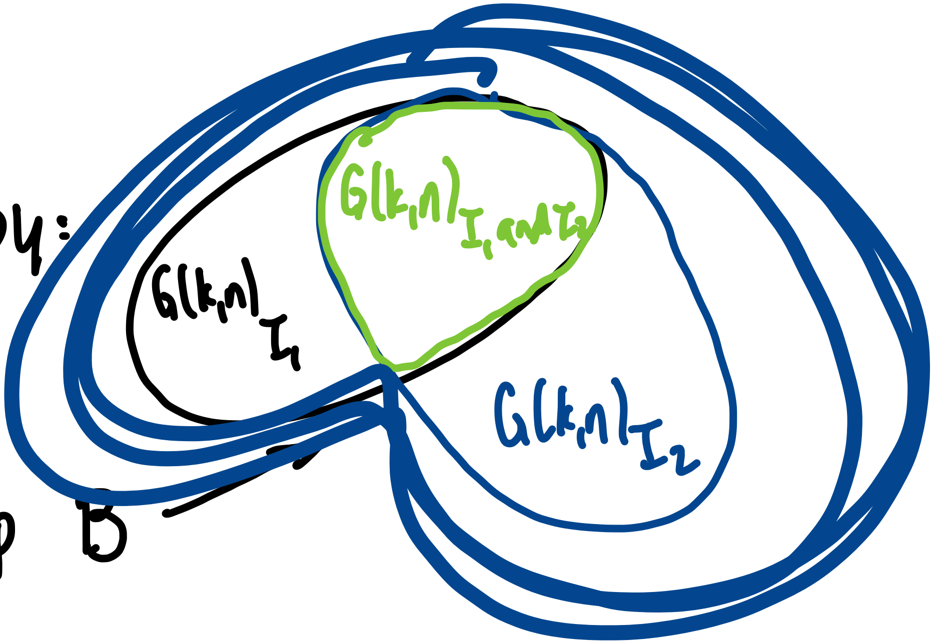
$$\begin{tikzcd}\mathcal{O}^{\oplus n} \arrow[r] \arrow[d, dash] & \mathcal{V} \arrow[r] & 0 \\ B & & \end{tikzcd}$$

such that $\mathcal{O}^{\oplus I_1} \xrightarrow{\sim} \mathcal{V}$

or $\mathcal{O}^{\oplus I_2} \xrightarrow{\sim} \mathcal{V}$.

Claim

This is representable! By:



Proof

Here is why. (i) Given a map B

tell me

$$\begin{tikzcd}\mathcal{O}^{\oplus n} \arrow[r] \arrow[d, dash] & \mathcal{V} \arrow[r] & 0 \\ B & & \end{tikzcd}$$

such that ...