Moduli Spaces in Algebraic Geometry

Math 245 A (winter 2022)

Jan. 3, 2022.

Intent: A second course in algebraic geometry, serious and rigorous, but I'd like to bring along as many people as possible.

(Should be okay if you are currently taking 216B)

course webpage:

math.stanford.edu/~vakil/22-245moduli

office hours: TBA. What could work?

(balcony? office? stroll? zoom?)

First part of the course:

What is a moduli space, really?

How do you show they exist (when they do)?

How do you show they don't exist (when they don't)?

(What do you do if they "almost exist"?)

Showing some exist:

The Hilbert scheme.

The Quot scheme.

The Isom scheme.

Huge transformative idea.

Later parts of the course: depends...

Today: Setting the stage.

Vague ideas we might want to make precise:

Circles in $\mathbb{R}^2$.

(x_0, y_0) center

r radius

$r > 0$

$x_0, y_0, r \in \mathbb{R}$

$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

"Parameter"

"obvious"

Scalene triangles

up to congruence

up to similarity

in $\mathbb{R}^2$

in $\mathbb{R}^2$ up to rotation and translation

vertices named?

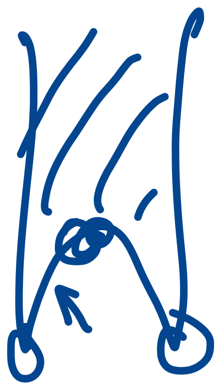
$1 < x < y$

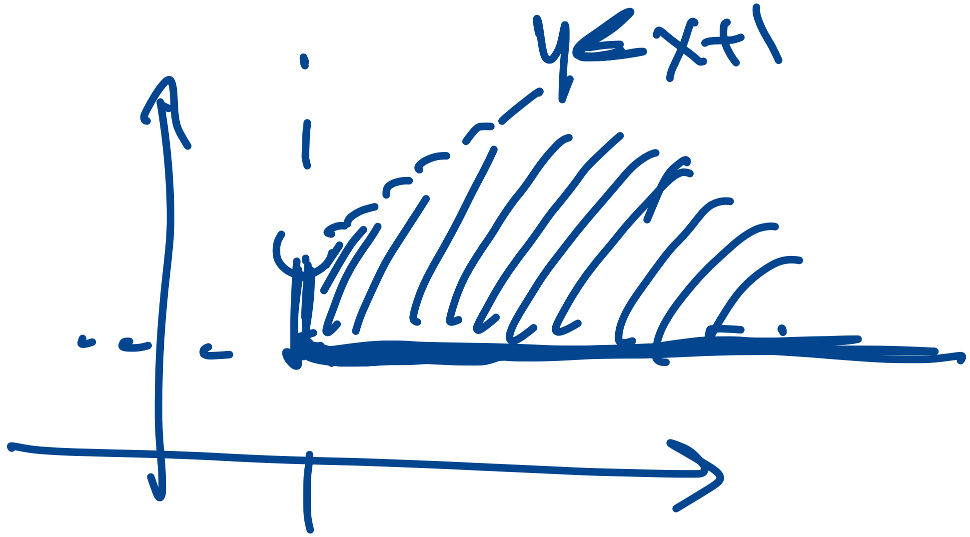
$\boxed{y < x+1}$

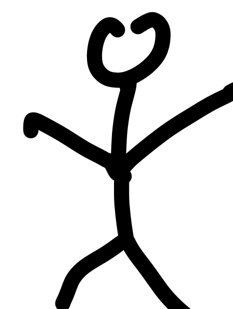
2-di:

Triangles ...

$1 \le x \le y$







Cubic surfaces in $\mathbb{CP}^3$.

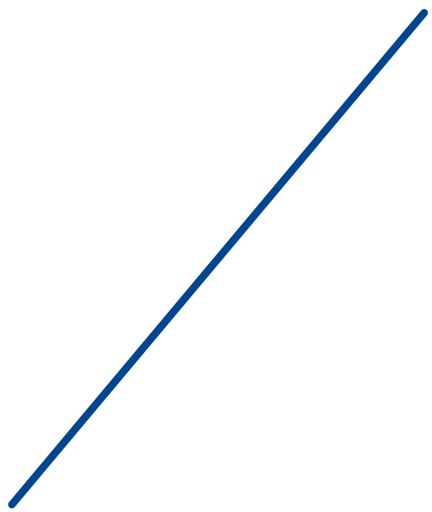
$$? \, x^3 + \, ? \, x^2 y + \cdots + \, ? \, y^3 + \cdots + \, ? \, z^3 = 0.$$

$$\boxed{\mathbb{CP}^9}$$

Algebraic hypersurfaces in $\mathbb{C}^3 = \mathbb{A}^3_{\mathbb{C}}$?

$$\hookrightarrow \; = \; ? + ?x + ?y + ?z + ?x^2 + \cdots$$

The space of lines in $\mathbb{R}^2$.



$y = mx + b$ "most lines" $\mathbb{R}^2$

$x = b'$ $\mathbb{R}$.

$x = m'y + b'$ "most lines" $\mathbb{R}^2$.

The Grassmannian:

$G(K,N)$ = space of "linear $\mathbb{P}^K$'s" in $\mathbb{P}^N$.

field? $/\mathbb{Z}$?

$K = k-1 \quad N = n-1$

$G(k,n)$ = space of k-dimensional linear subspaces of an n-dimensional vector space.

Thinking through The Grassmannian ...

$G(k,n)$ = space of k-dimensional linear subspaces of an n-dimensional vector space $\mathbb{C}^n$.

Answer 1: a unique name for any k-plane. $V \subset \mathbb{C}^n$

"normal form".

There is a unique basis for V of the form: "echelon form".

$$
k \text{ row vectors} \left\{ \begin{matrix} 0 & 0 & 0 & 0 & 1 & * & 0 & * & * & * & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & * & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & \cdots \end{matrix} \right. \quad ??
$$

"Most"
$$
\left\{ \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \;\; \boxed{\begin{matrix} * & * & * \end{matrix}} \right\}.
$$

$\mathbb{C}^{k(n-k)}$ ← manifold!

Down side: not very symmetric!

Why a manifold?

Answer 2: $U \subset (\mathbb{C}^n)^k$ linearly independent

$U/GL(k)$. "forget the basis"

Down side:

What is this?!

Answer 3: $GL(n)$ acts transitively on $\boxed{G(k,n).}$

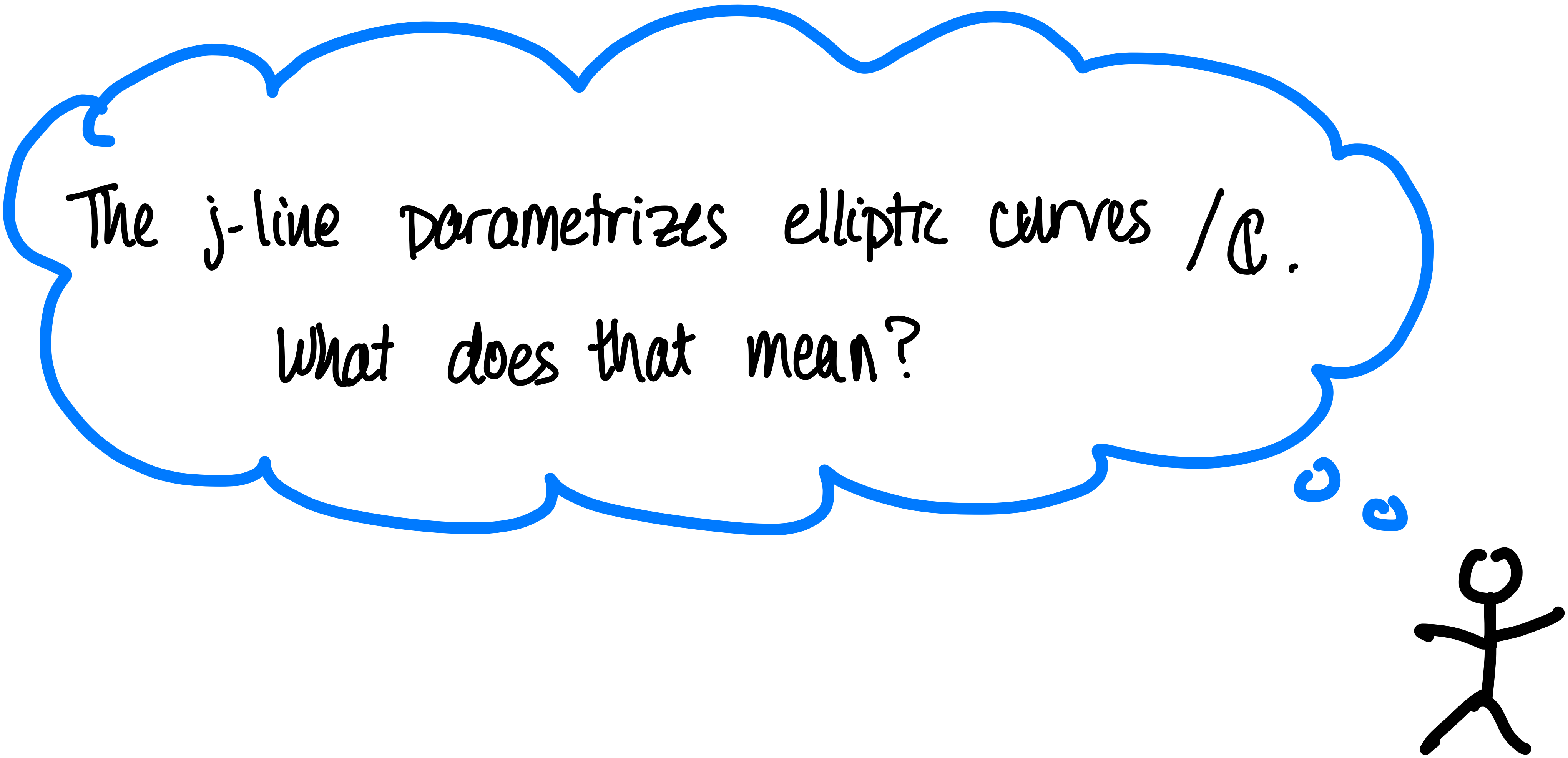
What is the stabilizer of the k-plane spanned by $e_1, \dots, e_k$?

Answer:

$$\begin{array}{cc}
\left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right] & \begin{array}{l} \}\, k \\ \}\, n-k \end{array} \\
\begin{array}{cc} \underbrace{\quad}_{k} & \underbrace{\qquad}_{n-k} \end{array} &
\end{array}$$

"maximal parabolic subgroup $P \subset GL(n)$"

We could believe that there is an "orbit stabilizer theorem": $G(k,n) = \boxed{GL(n)/P}$


The j-line parametrizes elliptic curves $/\mathbb{C}$.
What does that mean?

Last example: $\boxed{\mathcal{M}_3^a}$ "asymmetric genus 3 curves / Riemann surfaces"

no nontrivial automorphisms.

definition only for here

(scalene triangle = asymmetric triangle)

There is a 6-dimensional smooth complex variety (complex manifold) $\mathcal{M}_3^a$.

~~Why?~~

What does this even mean?

What <u>is</u> a moduli space?

What right do we have to give a manifold structure to $G(k,n)$ or $\mathbb{P}^n$?

MAIN IDEA OF TODAY (Grothendieck)

$\mathcal{M}$ is a moduli space for something

$\hookrightarrow$ in the world we are living in.

e.g. schemes or varieties if:

$X \longrightarrow$ Moduli space

(Schemes) := the category of schemes

(Varieties) etc.

There is a "natural" bijection between "nice" families of our objects, parametrized by B, and maps $B \to \mathcal{M}$.

soon figure that out

$$\left\{ \begin{array}{c} F \\ \downarrow \\ B \end{array} \right\} \xleftrightarrow{\text{bijection}} \left\{ B \to \mathcal{M} \right\}$$

equivalence?!

B'

$$\begin{array}{ccc} B & & \mathrm{Hom}(B, \mathcal{M}) \\ \uparrow & & \downarrow \\ B' & & \mathrm{Hom}(B', \mathcal{M}). \end{array}$$

Let us unpack this further.

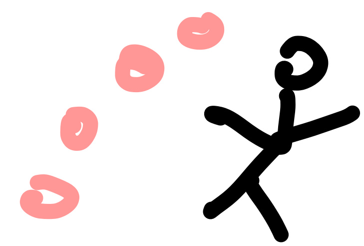
We can hope to have a notion of moduli space for the following kind of notion.

For every $B \in (\text{Schemes})$

a set of (nice) "families over B" $\mathcal{F}(B)$

that we know how to "pull back"

$$B' \longrightarrow B \qquad\qquad \mathcal{F}(B) \longrightarrow \mathcal{F}(B')$$



fancy: a contravariant functor from (Schemes) to (Sets).

Then we say $\mathcal{M}$ is a moduli space for our objects if we have an isomorphism of functors

$$\mathcal{F} \xrightarrow{\sim} \mathrm{Hom}(\cdot, \mathcal{M}).$$

~~(Better: a moduli space for $\mathcal{F}$ is ...)~~

Translation: the moduli functor is a representable functor.

sounds fancy, but no new content!

To avoid confusion,

some might call $\mathcal{M}$ a "fine" moduli space, to

distinguish it from sad approximations of this notion

("coarse" moduli space, "good" moduli space)

$\hookrightarrow$ defined by

Jarod Alper

The magic of $\boxed{\text{Yoneda's Lemma}}$:

If a moduli space exists, then it is unique up to unique isomorphism.

This gives the right to say that something is the moduli space for some sort of thing.

HOMEWORK: Think this through.

Write it down if you have not done so before.

So to ask for a moduli space, we need something to take moduli spaces *of*.

"Moduli functor" = "moduli problem"

old-fashioned

Contravariant functor (Schemes) $\to$ (Sets).

Translation: tell me the families of your objects.

Examples of moduli functors:

$B \mapsto$

closed subschemes of $\mathbb{P}^n_B$

$$\downarrow$$

B

$B \mapsto$

Fix a curve C.

Vector bundles on

$B \times C$

$$\downarrow$$

B

$B \mapsto$

C

$$\downarrow$$

B

fibers are smooth genus g curves, or abelian varieties

projective

smooth

rel. dim 1

fibers to be (geom. irred.) curves of genus g.

Jacobian:

up to iso?

"nice"?

The universal family

If the moduli space $\mathcal{M}$ exists,

it has a "universal family" over it,

from which every other family is pulled back.

$$\begin{tikzcd} \mathcal{O} \arrow[r] \arrow[d] & \mathcal{U}_{\mathcal{M}} \arrow[d] \\ B \arrow[r] & \mathcal{M} \end{tikzcd}$$

$$\text{maps}\left(B \to \mathcal{M}\right) \sim \begin{array}{c} \text{"mod family"} \\ \downarrow \\ B \end{array}$$

$$\text{Map}\left(\mathcal{M} \xrightarrow{id} \mathcal{M}\right) \sim \begin{array}{c} \mathcal{U}_{\mathcal{M}} \\ \text{"mod family"} \\ \downarrow \\ \mathcal{M} \end{array}$$

Examples of moduli spaces

(i.e., moduli functors that are representable)

Dumb but true:

Maps to a fixed scheme Y.

$$B \longrightarrow \mathrm{Maps}(B, Y)$$

Less dumb:

The moduli space of functions.

$$B \mapsto \mathcal{O}(B).$$