18.034 PRACTICE MIDTERM 2

Midterm 2 will take place Friday April 14 in class (no calculators). There will be 6 questions. Office hours the week beforehand will be Wednesday 11-12 and Thursday 3:30-6.

Topics.

Look at the course webpage to see what we've covered, and for references to the text. In the book, we've now covered Chapters 1–4, and 7.1–7.3. The Contraction Mapping Theorem material isn't in the book, so you'll have to rely on your notes.

Existence and uniqueness theorems for higher-order linear equations, e.g. Theorem 4.1.1.

Wronskians. Abel's theorem. Especially in order 2: using them to find all homogeneous solutions given one, using them to find a general solution given the homogeneous solutions (variation of parameters).

The Contraction Mapping Theorem. The Existence and Uniqueness Theorem (of BD 2.11). Lipschitz conditions, etc.

Linear independence, eigenvalues, eigenvectors.

Anything on the problem sets is fair game, as is material covered before the first midterm.

Sample questions.

Answers will be available at some point in the week before the midterm, on the course webpage.

1. Suppose you have four solutions to the fourth-order equation $ty'''' + y''' + \sin(t)y'' + y = 0$. What can the Wronskian be?

2. A trick question: Calculate the Wronskian $W(t^2 + t, t^2, t)$.

3. Suppose $y_1(t)$ and $y_2(t)$ are two functions on $I = (0, \infty)$ such that $W(y_1, y_2)(t) = t - 1$. Can they both be solutions to a differential equation of the form

$$y'' + a_1(t)y' + a_0 y = 0$$

on I, where $a_1(t)$ and $a_0(t)$ are continuous on I?

Date: April 10, 2000.

4. Consider the differential equation

$$(1-t)y'' + ty' - y = 0$$

on the interval $1 < t < \infty$. Notice that $y_1(t) = t$ is a solution; find a second solution $y_2(t)$ as follows. Suppose you have found y_2; calculate the Wronskian $W(y_1, y_2)$. Use this, and $y_1 = t$, to find a differential equation satisfied by y_2. Then find some y_2 by solving the differential equation. (As a check, you may want to plug y_2 back in the original differential equation.) *Hint:* $\int \frac{1-t}{t^2} e^t dt = -e^t/t + C$.

5. Use "variation of parameters" to solve the problem $y'' + y = f(x)$ for a general continuous function $f(x)$. (The answer is $\int_0^x \sin(x-s)f(s)ds$.)

Hint: Note that $\phi_1 = \cos(x)$ and $\phi_2 = \sin(x)$ are linearly independent solutions to the homogeneous version of the equation. Look for a solution of the form $\phi(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$.

6. State the Contraction Mapping Theorem.

7. Suppose you have an ancient twentieth century calculator that only has the operations $+$, $-$, $\times$, $/$. Use the Contraction Mapping Theorem to explain how to get a good approximation to $\sqrt{3}$.

8. State the Existence and Uniqueness Theorem for differential equations of the form $y' = f(x, y)$.

9. Transform the initial value problem $y' = 1 - y^3$, $y(-1) = 3$ into an equivalent problem with the initial point at the origin.

10. Without using properties of the exponential function, solve the following problem. Suppose $z(t)$ is a solution to the differential equation $y' = y$, that is defined on the real line, such that $y(0) = 1$. Prove that $z(a)z(b) = z(a+b)$. *Hint:* Consider the function $w(t) = z(t+a)/z(a)$, and show that is is the solution to the initial value problem $y' = y$ with initial condition $y(0) = 1$.

11. Calculate

$$\begin{pmatrix} 5 & 1 \\ 4 & 5 \end{pmatrix}^{100} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$