18.034 PRACTICE MIDTERM 1

Topics.

Look at the course webpage to see what we've covered (and for references to the text). In the book, we've now covered: Chapters 1, 2 (except 2.11), 3 (except 3.3 and 3.7), 4 (except 4.4).

Basic definitions: ODE, PDE, order, solution, linear ODE, direction field, integral curve. (I won't ask you definitional questions, but you'll need to know them in order to answer the problems.)

First-order linear ODEs: existence and uniqueness theorem. An example where uniqueness fails.

Integrating factors. Existence and uniqueness in this case.

Separable equations.

Autonomous equations (phase line, stable equilibria, unstable equilibria).

Exact equations. The "closed" condition. Understand equations that are closed but not exact.

Equations $y' = f(x, y)$ where $f(x, y)$ is purely a function of y/x; substitute $v = y/x$.

Homogeneous and nonhomogeneous ODEs with constant coefficients. (The language of linear operators. The exponential shift law.)

Recurrences. Know how to solve them.

Complex numbers.

The special case of second-order nonhomogeneous ODEs with constant coefficients, and the various behaviors that can come up.

Anything on the problem sets are fair game. But you won't need to memorize how to solve specialized sorts of equations, such as Ricatti equations; just know how to solve them if you are given the substitution.

Sample questions.

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Answers are available on the course webpage.

1. Solve the differential equation $y' + 2ty = 2t \sin t + \cos t$ with the initial condition $y(0) = 1$. What is the long-term behavior of the solution?

2. Consider the differential equation $y' = 9y^2 - 1$. This is an autonomous equation. Suppose $y(0) = 0$. What happens for large t? What are the equilibrium values of y? Are the stable or unstable?

3. Consider the differential equation $y' = 1/(9y^2 - 1)$, with initial condition $y(0) = 0$. Show that the solution does not exist for all time.

4. $(x_0, x_1, x_2, \dots)$ is a sequence of numbers satisfying $6x_n = 5x_{n-1} - x_{n-2}$, $x_0 = 0$, $x_1 = 1/6$. Find a formula for x_n. Show that for *any* initial conditions, $\lim_{n\to\infty} x_n = 0$.

5. Rewrite $\cos 3t - \sqrt{3} \sin 3t$ as $\cos(\omega t - \delta)$ for appropriate ω, δ.

6. Let z_1 be the complex number $a + bi$, and z_2 the complex number $c + di$. Calculate: $|z_1|$, $|z_2|$, and $|z_1 z_2|$. Prove that $|z_1||z_2| = |z_1 z_2|$. What do you notice? Are you surprised?

7. Complete the statement of the Existence and Uniqueness Theorem for differential equations of the form $y' = f(t, y)$, $y(0) = 0$:

If ——————————— are continuous in a rectangle $|t| \le a$, $|y| \le b$, then there is some interval $|t| \le h \le a$ in which there exists a unique solution $y = \phi(t)$ of the differential equation.

Give an example of a differential equation of the form above for which uniqueness does not hold.

8. Find the general solution to the differential equation $y'' + 6y' + 10y = 0$. Sketch a typical solution. Is it underdamped, critically damped, or overdamped? Change the damping so that the answer to the previous question would be different, and solve the changed differential equation.

9. Now consider the differential equation $y'' + y' + y = \cos \omega t$. What is the frequency of the transient solutions? Find the amplitude of the steady-state solution (in terms of ω). Find ω so that the amplitude is as large as possible.

10. Solve the differential equation $D(D + 1)y = e^t$ as follows. Let $z = (D + 1)y$, find a differential equation for z, and find the general solution. Then using this general value of z, find y by solving the differential equation $(D + 1)y = z$.

11. Recall the Bernoulli-type equation, a differential equation of the form $y' + p(t)y = q(t)y^n$, where $n = 0$ or 1. If you were given some nice $p(t)$ and $q(t)$, show how you would solve the equation using the substitution $v = y^{1-n}$.