SKETCHES OF SOLUTIONS TO 18.024 PRACTICE QUIZ II

1. (16 points) Consider the curve given in polar coordinates by $r = e^{-t}$, $\theta = t$ for $0 \leq t \leq 2M\pi$ (M a positive integer).

(a) Sketch this curve when $M = 2$.
(b) Find the length of this curve for general M. What happens as M becomes large?

Solution. (a) Your sketch should wind inwards counterclockwise around the origin twice. (b) Use the formula for $\vec{v}$ (given at the top of the practice quiz) to get

$$\int_0^{2M\pi} \sqrt{(r)^2 + (r')^2} dt = \int_0^{2M\pi} \sqrt{e^{-2t} + e^{-2t}} dt = \sqrt{2}\left(1 - e^{-2M\pi}\right).$$

As $M \to \infty$, the length goes to $\sqrt{2}$.

2. (16 points) A particle moves along a curve C in space. Its acceleration vector has constant length 3 and its speed at time $t \geq 0$ is $1/(1+2t)$. Find the curvature of the curve in terms of t.

Solution. Use the formula

$$\vec{a} = v'\vec{T} + \kappa v^2 \vec{N}$$

(the second equation given at the top of the practice quiz) from which

$$||\vec{a}||^2 = (v')^2 + \kappa^2 v^4.$$

This gives

$$\kappa = \sqrt{36t^2 + 36t + 5}.$$

3. (20 points)

(a) Complete the definition. A vector-valued function

$$\vec{f} : S \to \mathbb{R}^3$$

where $S \subset \mathbb{R}^2$ contains a ball $B(\vec{a}; r)$ of radius r around $\vec{a} \in \mathbb{R}^2$ is said to be *differentiable at* $\vec{a}$ if for all $\vec{v} \in$ [blank],

$$\vec{f}(\vec{a} + \vec{v}) = \vec{f}(\vec{a}) + T_a \vec{v} + |\vec{v}| E_{\vec{a}}(\vec{v})$$

where [blank]. (Hint: the second blank is a fact about the function $E_{\vec{a}}$.)
(b) Let $f(x, y)$ be a function defined in $\mathbb{R}^2$ (the plane). Answer the following questions "yes" or "no" (+3 points for each correct answer, −3 for each incorrect answer).

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(i) Suppose that $D_1 f$ and $D_2 f$ exist at $(0,0)$. Does it follow that f is continuous at $(0,0)$? **Y/N** Does it follow that the functions $g(t) = f(t,0)$ and $h(t) = f(0,t)$ are continuous at $t = 0$? **Y/N**

(ii) Suppose that $D_1 f$ and $D_2 f$ exist in a neighborhood of $(0,0)$ and are continuous at $(0,0)$. Does it follow that f is continuous at $(0,0)$? **Y/N** Does it follow that $f'(\vec{0}; \vec{y})$ exists for all $\vec{y}$? **Y/N**

Solution. (a) $\vec{v} \in B(\vec{0}; r)$ (*not* $B(\vec{a}; r)$ — do you see why?); $E_{\vec{a}}(\vec{v}) \to 0$ as $\vec{v} \to \vec{0}$ (remember to write $\vec{0}$ not 0 for the zero vector). (b) (i) NY (b) YY.

4. (16 points)

(a) Suppose $f(x,y,z)$ is a differentiable scalar-valued function such that $f(1,1,1) = 2$, and the $\vec{\nabla} f(1,1,1) = (3,4,5)$. Find the (Cartesian) equation of the tangent plane to the level surface $f(x,y,z) = 2$ at $(x,y,z) = (1,1,1)$.

(b) Suppose $f(x,y)$ is a differentiable scalar-valued function such that $f(1,1) = 2$, and $\vec{\nabla} f(1,1) = (3,4)$. Find the (Cartesian) equation of the tangent plane to the graph of f (i.e. $z = f(x,y)$) when $(x,y) = (1,1)$.

(c) Show that $f(x,y) = (\sin x)(\sin y)$ has a critical point (i.e. the gradient is $\vec{0}$) at $(x,y) = (0,0)$. Does f have a minimum, maximum, or saddle point here?

Solution. For (a) and (b), use the gradient. (a) $3(x-1)+4(y-1)+5(z-1) = 0$, or $3x+4y+5z = 12$ (either is acceptable of course). (b) $3(x-1)+4(y-1)-(z-2) = 0$ or $(z-2) = 3(x-1)+4(y-1)$ or $z = 3x+4y-5$. (c) Saddle points (using second derivative test). (You can also see this geometrically: which direction can you walk from $(0,0)$ so f decreases? Increases?)

5. (16 points) Given a differentiable function $F(u,v)$, consider the composite function $f(x,y) = F(3x-y, 2x-y)$. Find $\frac{\partial f}{\partial x}(1,1)$ if $D_1 F(1,1) = -4$, $D_1 F(2,1) = 7$, $D_2 F(1,1) = 3$, $D_2 F(2,1) = -3$.

Solution. Use the chain rule. Let $u = 3x - y$, $v = 2x - y$. Then

$$\begin{aligned} \frac{\partial f}{\partial x}(1,1) &= \frac{\partial F}{\partial u}(u=2, v=1)\frac{\partial u}{\partial x}(x=1, y=1) + \frac{\partial F}{\partial v}(u=2, v=1)\frac{\partial v}{\partial x}(x=1, y=1) \\ &= 7 \times 3 + (-3) \times 2 \\ &= 15. \end{aligned}$$

(Notice the red herring in the problem: we never use $D_i F(1,1)$!)

6. (16 points) The equation $x^2 + z^3 + yz = 3$ defines z implicitly as a function of x and y, say $z = f(x,y)$. Find $\frac{\partial f}{\partial x}$ in terms of x, y, and z. Find $\frac{\partial^2 f}{\partial x^2}$ in terms of x, y, z, and $\frac{\partial f}{\partial x}$.

Solution. Use implicit differentiation. Differentiate $x^2 + f(x,y)^3 + yf(x,y) = 3$ with respect to x to get

$$2x + 3f^2 f_x + y f_x = 0$$

from which

$$f_x = \frac{-2x}{3f^2 + y} = \frac{-2x}{3z^2 + y}.$$

Differentiate this again with respect to x to get

$$f_{xx} = \frac{-2(3f^2 + y) - (-2x)(6ff_x)}{(3f^2 + y)^2} = \frac{-2(3z^2 + y) - (-2x)(6zf_x)}{(3z^2 + y)^2}.$$

If you wanted, you could substitute the formula for f_x (in terms of x, y, and z) into this one, but it would be ugly, and nothing much would be gained.)