Algebraicity of Persistent Homology

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Let $f_1, \ldots, f_s \in \mathbb{Q}[x_1, \ldots, x_n]$ and let $X = V(f_1, \ldots, f_s)$ be the real points in the variety V.

Definition. The *true persistent homology of a variety* X *at parameter* ϵ is the homology of its ϵ-neighborhood.

Theorem(H.-W.'18). The values of the persistence parameter ϵ at which a bar in the true barcode appears or disappears are real numbers algebraic over $\mathbb{Q}$.

Proof idea. Use Hardt's theorem from real algebraic geometry to obtain a semi-algebraic trivialization.

Definition. The *ϵ-offset hypersurface* $\mathcal{O}_\epsilon(X)$ is the envelope of ϵ-balls centered at a point x on the variety. As x varies, we obtain the *ϵ-offset correspondence* $\mathcal{OC}_\epsilon(X)$.

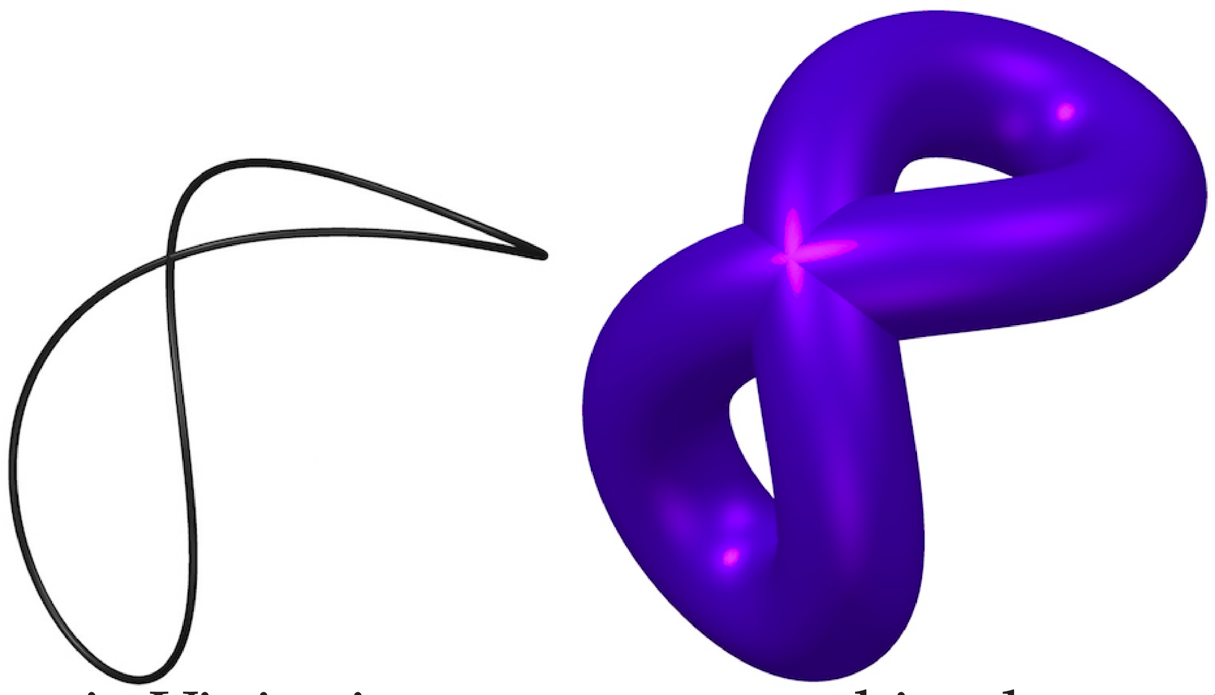

Figure 1: The quartic Viviani space curve and its degree 10 offset surface.

Consider the projection $\mathcal{OC}_\epsilon(X) \rightarrow \mathcal{O}_\epsilon(X)$. The branch locus of this projection consists of points y for which the ϵ-ball centered around y is at least doubly tangent to the variety X. We denote the closure of the union of all branch loci over ϵ in $\mathbb{R}_{\geq 0}$ by $B(X, X)$, called the *bisector hypersurface of the variety* X.

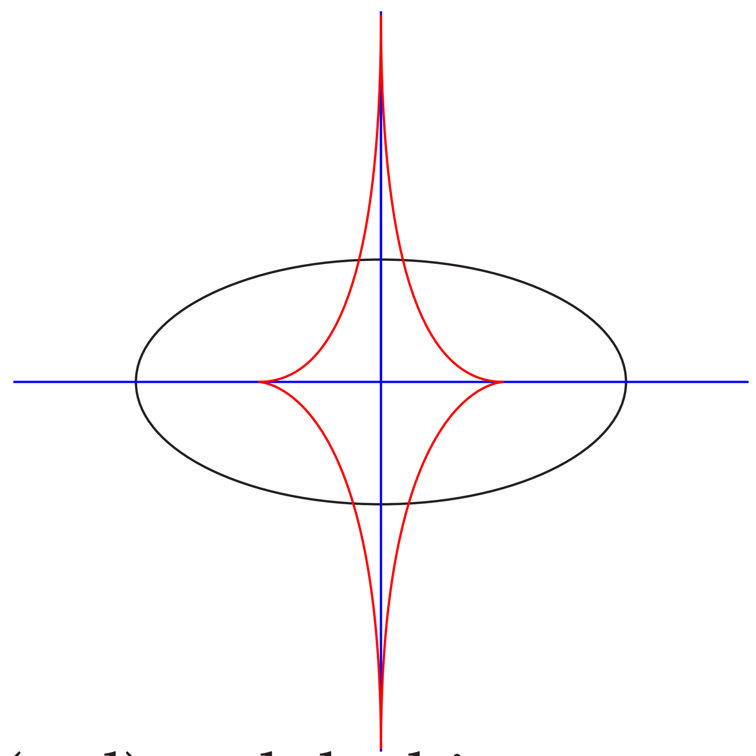

Figure 2: The evolute (red) and the bisector curve (blue) of an ellipse.

Theorem(H.-W.'18). (*Geometric interpretation of endpoints in barcode.*) Let $X \subset \mathbb{R}^n$ be a hypersurface. Let $J = \{[\delta_l, \epsilon_l] | l \in \{1, \ldots, m\}\}$ be the set of intervals in the barcode for the top dimensional Betti number. Then each interval endpoint ϵ_l corresponds to a point y_l on the bisector hypersurface $B(X, X)$ which is an isolated real point in $\mathcal{O}_{\epsilon_l}(X)$. Furthermore, y_l is the limit of a sequence of centers of hyperballs contained in the complement of $\mathcal{O}_\epsilon(X)$ as $\epsilon \rightarrow \epsilon_l$.

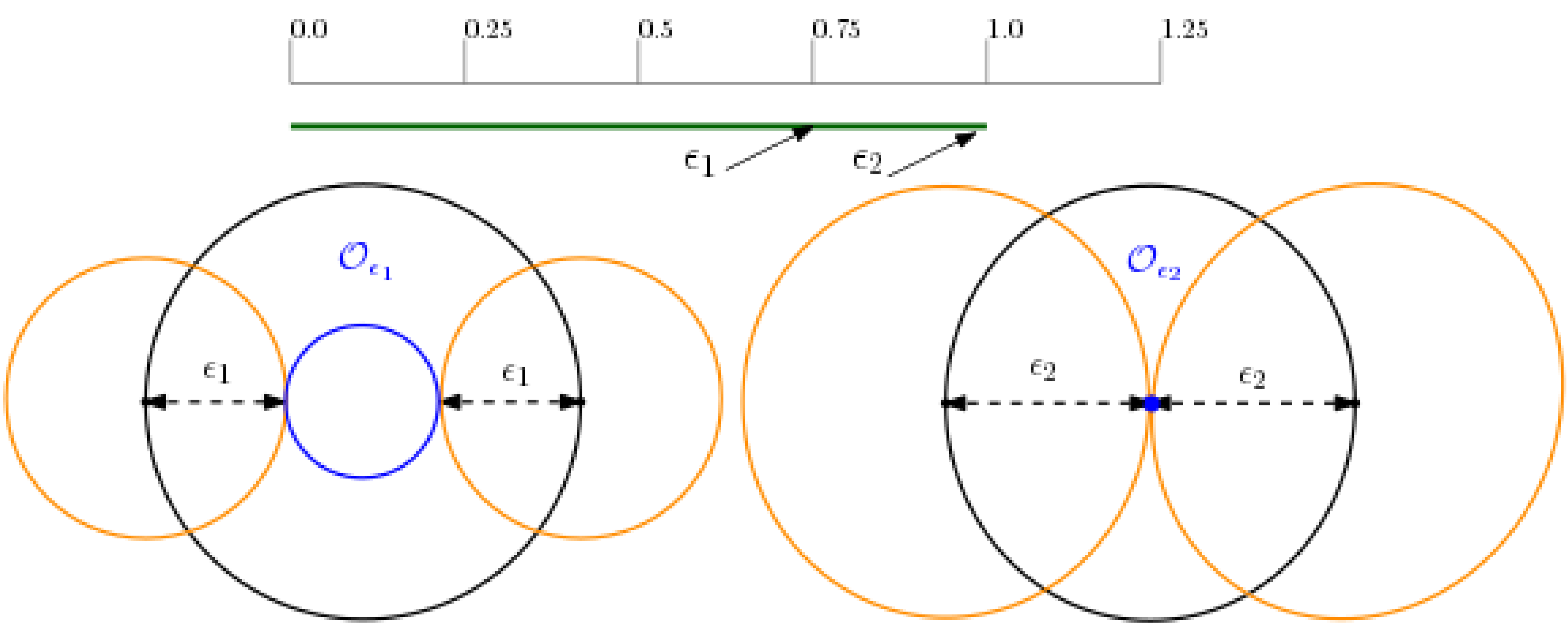

Figure 3: These pictures illustrate how the offset variety provides a geometric interpretation of the endpoints of a bar. The black circle is the variety X and the orange circles are ϵ- balls around X. When ϵ reaches the radius of the black circle, the blue offset hypersurface $\mathcal{O}_\epsilon$ has an isolated real point.