Relative Trisections (just a reminder)

\[ X^4 = \text{compact, connected, orientable 4-mfld with nonempty boundary} \]

A \((g, k, p, b)\) relative trisection of \(X\) is a decomposition \(X = Y_1 \cup Y_2 \cup Y_3\) where

\[
\begin{align*}
\cdot Y_i & \cong \frac{1}{k_i} S' \times B^3 \quad \text{3D pieces are handlebodies} \\
\cdot Y_i \cap Y_{i+1} \text{ intersect only in their boundaries} \\
\cdot Y_i \cap Y_j \cong \frac{1}{k_i} S' \times D^2 \quad \text{double intersections are 3D handlebodies} \\
\cdot Y_1 \cap Y_2 \cap Y_3 & \cong \text{genus-}g \text{ surface with } b \text{ boundary components} \\
\cdot Y_i \cap \partial X^4 & \text{ is a product} \\
Y_i \cap \partial X^4 & = (Y_i \cap Y_{i-1}) \times I \\
(\Rightarrow & = (Y_i \cap Y_{i+1}) \times I \text{ with opposite orientation})
\end{align*}
\]

Analogous to trisection of closed manifold
By £
$B^4 \cong \Sigma_0 \cong D^2$
$H_\beta := \Sigma_2 \cap \Sigma_3 \cong B^4$
$(g, k; p, b) = (0, 1; 0, 1)$
relative trisection of $B^4$

$H_\alpha := \Sigma_1 \cap \Sigma_2 \cong B^3$

On $\partial B^4$:
$
\Sigma$ is the binding of open book

(Here, unknotted pages are disks)

$H_\alpha \cap \partial B^4 \cong D^2$
$
\Sigma_1 \cap \partial B^4 \cong D^2 \times I$
$= H_\alpha \times I$
Note $H_{\alpha} \cap X^4$ is a page of the open book.
(So are $H_{\beta} \cap X^4$ and $H_{\gamma} \cap X^4$).

We call $H_{\alpha} \cap X^4$ the $\alpha$-page
($H_{\beta} \cap X^4$= $\beta$-page, $H_{\gamma} \cap X^4$= $\gamma$-page.)

The $\alpha$-page is parallel to $\Sigma$ compressed along $\alpha$.

$\Sigma$, $\alpha$, $\beta$, $\gamma$
Point of the open book criteria: So that diagrams still define a manifold.

Diagram for relative trisection $X = Y_1 \cup Y_2 \cup Y_3$

$$(E^b_g, \alpha, \beta, \gamma)$$ where

$H_\alpha := Y_1 \cup Y_2$ is obtained from $E^b_g \times I$ by attaching 3D 2-handles to $\alpha \times I$

$H_\beta := \beta \times I$

$H_\gamma := \gamma \times I$

e.g. 

![Diagram](image-url)
To recover $X^4$ from $(\Sigma, \alpha, \beta, \gamma)$:

Now just need to glue in $Y_1, Y_2, Y_3$. In closed case, Landenbach-Poenaru lets us do this without making a choice. Now, some of $DY_i$ isn’t in this picture, so we specify it by saying $Y_1, n\partial X^4$ is a product. Now we can use LP.

Glue $Y_1$ using LP to avoid any choices.

$H_\gamma \cap \partial X^4 \times I$

$= -H_\alpha \cap \partial X^4 \times I$
Theorem (Gay-Kirby '12)
(Castro-Gay-Pinzón-Caicedo '16)

Every (cpt connected orientable) $X^4$ admits a relative trisection.

Theorem (Gay-Kirby '12)
(Castro '16)
For any open book $\mathcal{O}$ on $\mathbb{R} \times X^4$, there exists a relative trisection of $X^4$ inducing $\mathcal{O}$.

So in general, relative trisections of $X^4$ are not related by interior stabilization

$3 \mathcal{O} \to \mathcal{O}$ (connect-summing with a genus-1 trisection of $S^4$)

because this doesn't change the open book induced on $\mathbb{R} \times X^4$.

Thm (Gay-Kirby 2012) If relative trisections $T_1, T_2$ of $X^4$ induce isotopic open books on $\mathbb{R} \times X^4$, then they are related by interior stabilization.
Relative stabilization

Move introduced by Castro '16, studied diagramatically by Castro-Gay-Pinzón-Carvacho '16

Effect on open book: plumbing on Hopf band. (Sign depends on handedness of $\partial X$ in the picture.)

Thm (Castro '16) (Consequence of above move + Giroux-Goodman)

If $\partial X \cong \mathbb{Z} \times S^3$, then any two relative trisections of $X$ are related by a sequence of interior and relative stabilizations/destabilizations.
Key theorem (Piergallini-Zuddas 2018)

Any two open books of closed, compact, orientable, connected $M^3$ become isotopic after some number of

- Hopf stabilization/destabilization
- "dU" and inverse

Open book

This disk in page is fixed pointwise by monodromy

Puncture twice and add Dehn twists about new boundary to monodromy
Corresponding trisection move (Castro-Islambouli-M-Tomova)

This disk is in $\Sigma \setminus (\alpha \cup \beta \cup \delta)$

relative double twist

ex

$D^2$
Check: What does this do to 4-mfd? (Kim-M 2018)

Recall that by standardizing \((\alpha, \beta)\), a relative trisection can yield a Kirby diagram.

- parallel \(\alpha, \beta\) and cut arcs \(\Rightarrow\) 1-handles for \(\alpha, \beta\) page
- \(\gamma\) dual to \(\beta\) \(\Rightarrow\) 2-handles
- parallel \(\alpha, \gamma\) \(\Rightarrow\) 3-handles
The new arcs are parallel away from the stabilization.
I safed so this is really a trisection move

Same 4-mfd (so this is really a trisection move)
If we tried to relative double twist but allowed $\alpha, \beta, \gamma$ to separate the two punctures, the manifold could change.

$$\mathbb{CP}^2 \setminus B^4$$

$$\frac{1}{1} \quad 1$$

$$\frac{1}{1} \quad \frac{1}{1} = S^2 \times D^2$$
What does relative double twist do to boundary open book?

Recall: Monodromy of open book can be computed by algorithm (Castro-Gay-Pintón-Caicedo '16)

(say $\alpha, \beta$ standard)

(Might also have to slide $\beta$ curves)

Step 1. Cut arcs a far a page.

Step 2. Slide over $\beta$ until disjoint from $\gamma$. 

\[ \text{Diagram with arrows indicating cutting and sliding.} \]
Step 3. Slide over $\gamma$ until disjoint from $\alpha$.

(Might also have to slide $\gamma$ curves)

Page (Hand X)

\[ a \rightarrow a' \]

$a'$ is the image of $a$ under monodromy

After relative double twist,

Can perform algorithm with same 1 slides away from stabilization.

The new arcs are parallel away from the stabilization.
near stabilization, have:

slide over $\beta$

slide over $\gamma$

$\alpha$ page
.\. Effect of relative double twist on boundary open book is \( \mathbb{Z} / 2 \mathbb{Z} \!
\)

(We punctured the \( \alpha \) page twice and changed the monodromy by adding opposite Dehn twists around the new boundaries)

This disk in page is fixed by monodromy \( \mathbb{Z} / 2 \mathbb{Z} \)
Moreover, any $\mathcal{U}$ move can be achieved by a relative double twist.

**Proof**

Let $\Delta$ be a disk in a page fixed by monodromy.

In trisection diagram,

$$\Delta \rightarrow \text{disk disjoint from } \alpha \text{ curves}.$$ 

Since $\Delta$ is fixed by monodromy, performing $\mathcal{U}$ move at any two points (choices of punctures) in $\Delta$ achieves the same open book. So just shrink $\Delta$ to avoid $\beta, \gamma$ curves and do relative double twist.
Consequence (Castro-Islambudzi-M-Tomova)

If $T_1$ and $T_2$ are relative trisections of a compact, connected, orientable 4-manifold $X^4$ ($\partial X^4 \neq \emptyset$) then $T_1$ and $T_2$ become isotopic after a finite number of

- interior stabilization

\[
\begin{array}{c}
\text{1} \\
\rightarrow \\
\text{any permutation of red, green, blue}
\end{array}
\]

- boundary stabilization

\[
\begin{array}{c}
\text{can be left or right twist} \\
\rightarrow \\
\text{and inverse}
\end{array}
\]

- relative double twist

\[
\begin{array}{c}
\text{and inverse}
\end{array}
\]
By Piergallini–Zuddas, there exist relative trisections $T_i$, $T_2$ such that relative stabilizations, double twists, and inverses induce isotopic open books on $\mathbb{R}^4$.

Then by Gay–Kirby, $T_i$ and $T_2$ have a common stabilization $T$.