This paper is a continuation of the work in [21]. We give a version of Seifert’s algorithm for surfaces in $S^4$ using triplane diagrams. We show how to explicitly produce a Heegaard diagram of a Seifert surface for any 2-knot $S$ given a triplane diagram of $S$. We also give some classification results, showing that if a surface $\Sigma$ in $S^4$ admits a $b$-bridge trisection with at least $b - 1$ disks in one sector, then $\Sigma$ is unknotted. This answers a question of the second and fourth co-authors from 2017.

We construct pairs of same-genus Seifert surfaces in $S^3$ with common boundary that do not become isotopic in $B^4$ when their interiors are pushed into $B^4$. Almost all examples of same-genus but non-isotopic Seifert surfaces for a knot in the literature do become isotopic when pushed into $B^4$ (we explain this more concretely in the introduction of the paper). The question of whether this is a general phenomenon was suggested by Livingston in 1982; this paper answers that question. (Note that this question of Livingston was a low-dimensional motivation for [20], which motivated my continued interest in the problem solved in this paper.)

We give examples where the surfaces are not topologically isotopic in $B^4$, with obstruction the intersection form of 2-fold covers of $B^4$ branched along the surfaces. We also give examples that are topologically but not smoothly isotopic, with smooth obstruction the associated cobordism maps on Khovanov homology (and topological isotopy due to work of Conway and Powell).

We show that for every $g \geq 2$, there exist genus-$g$ 3-dimensional solids $H_1$ and $H_2$ smoothly embedded in $S^4$ so that the following are true.

1. $H_1$ and $H_2$ have the same boundary.
2. $H_1$ and $H_2$ are homeomorphic rel boundary.
3. $H_1$ and $H_2$ are not isotopic rel boundary even through a locally flat (rather than smooth) isotopy, even when their interiors are pushed into $B^3$.

This is a strong resolution to the case $g \geq 2$ of a conjecture of Budney–Gabai, who posited that for all $g \geq 0$ there exist genus-$g$ solids $H_1$ and $H_2$ smoothly embedded in $S^4$ that have the same boundary and are homeomorphic rel boundary but not smoothly isotopic rel boundary in $S^4$. Budney–Gabai proved this was true for $g = 0$; we prove this is true in an extremely strong sense (since $H_1$ and $H_2$ do not become isotopic when pushed into $B^3$) for $g \geq 2$. This last point is particularly surprising, since the following lower-dimensional question remained open at the time this paper was written: Does there exist a knot $K$ in $S^3$ bounding two genus-$g$ Seifert surfaces $F_1$ and $F_2$ so that $F_1$ and $F_2$ do not become (smoothly/locally flatly) isotopic rel boundary when their interiors are pushed into $B^4$? (Note: this question was answered in my later joint work with Hayden, Kim, Park and Sundberg [22].)

We show that if \( \Sigma \) is a surface smoothly immersed in a 4-manifold \( X \) with isolated tranverse self-intersections, then \( \Sigma \) can be described by a banded singular link inside a Kirby diagram for \( X \). We show that if \( \Sigma' \) is (isotopic/regularly homotopic/homotopic) to \( \Sigma \), then the diagrams for \( \Sigma \) and \( \Sigma' \) are related by a certain explicit finite list of moves. We accomplish this by restricting an ambient Morse function on \( X \) to the complement of \( \Sigma \) and investigating flow lines of this Morse function between boundary and interior critical points. As an application, we show that a surface immersed in a trisected manifold can be put into trisection bridge position uniquely up to standard moves. We also use these singular banded diagrams to reprove some classical theorems about surfaces in 4-manifolds diagrammatically. This paper extends the work in [4].


For any natural number \( n \geq 2 \), we construct \( n \)-component surfaces \( \Sigma_1, \Sigma_2 \) smoothly, properly embedded into \( B^4 \) with the property that each \( \Sigma_i \) is *Brunnian* (meaning that if you delete any one component of \( \Sigma_i \) you get a surface that is smoothly isotopic rel. boundary to a Seifert surface for an unlink) and \( \Sigma_1, \Sigma_2 \) are an *exotic pair* (meaning that \( \Sigma_1 \) and \( \Sigma_2 \) are topologically isotopic rel. boundary but are not smoothly equivalent).

This is an example of a very subtle form of exotic behavior. If you delete one component from each of \( \Sigma_1 \) and \( \Sigma_2 \), the resulting surfaces are smoothly isotopic rel. boundary. Thus, an operation transforming \( \Sigma_1 \) into \( \Sigma_2 \) must somehow make use of every component of \( \Sigma_1 \), rather than being a local operation. When \( n = 2 \), we obstruct sliceness using an adjunction inequality or tools from knot Floer homology, in two different constructions. We then give an argument via branched coverings to reduce exoticness of \((n+1)\)-component surfaces to \(n\)-component surfaces.

The surfaces \( \Sigma_1 \) and \( \Sigma_2 \) can be chosen to consist only of disks. However, if we allow \( \Sigma_1 \) and \( \Sigma_2 \) to have a positive-genus component, then we can extend the result to construct infinitely many surfaces \( \{\Sigma_m\}_{m \in \mathbb{N}} \) which are pairwise exotic. This argument makes use of the tools from [13].


We show that a nonorientable 4-dimensional manifold built without 3- or 4-handles admits a Lefschetz fibration over the disk. In the relative nonorientable setting, regular fibers of a Lefschetz fibration are nonorientable surfaces with boundary. As a corollary, we obtain a 4-dimensional proof of the fact that every nonorientable closed 3-manifold admits an open book decomposition, which was first proved by Berstein and Edmonds using branched coverings. This paper was motivated by the work in [16].


We show that several classical theorems are true in the non-orientable setting, including Waldhausen’s theorem (i.e. that Heegaard splittings of \( \mathbb{#}_g S^2 \times S^1 \) are standard) and Lautenbach-Poenaru’s theorem (i.e. that any diffeomorphism of \( \partial(\mathbb{#}_k B^3 \times S^1) \) extends to a diffeomorphism of \( \mathbb{#}_k B^3 \times S^1 \)). This implies that Kirby diagrams and trisection diagrams of closed non-orientable 4-manifolds exist and are unique up to standard moves. Without a non-orientable Lautenbach-Poenaru theorem, one must explicitly describe attaching regions of 4-dimensional 3-handles in any surgery diagram, so this greatly improves our ability to describe non-orientable manifolds. We also describe how to use trisections to take covers and to describe embedded surfaces (as in the orientable case). In the case of Cappell-Shaneson homotopy 4-spheres (which are double covers of exotic \( \mathbb{R}^4 \’s \)), this yields potentially nonstandard trisections of standard \( S^4 \) along with trisections of potential exotic \( S^4 \’s \) that may be easier to analyze via trisection diagrams than with previously existing methods.
We prove a related statement: A knot is handle-ribbon if and only if it admits an R-link derivative; i.e. an n-component derivative L with the property that zero-framed surgery on L yields #^n(S^1 × S^3). We also show that K bounds a handle-ribbon disk D in a homotopy 4-ball B if and only if the 3-manifold obtained by zero-surgery on K admits a singular fibration that extends over handlebodies in B \ ν(D), generalizing a classical theorem of Casson and Gordon about fibered homotopy-ribbon knots to all handle-ribbon knots.

The stable Kauffman conjecture posits that a knot in S^3 is slice if and only if it admits a slice derivative. We prove a related statement: A knot is handle-ribbon if and only if it admits an R-link derivative; i.e. an n-component derivative L with the property that zero-framed surgery on L yields #^n(S^1 × S^3). We also show that K bounds a handle-ribbon disk D in a homotopy 4-ball B if and only if the 3-manifold obtained by zero-surgery on K admits a singular fibration that extends over handlebodies in B \ ν(D), generalizing a classical theorem of Casson and Gordon about fibered homotopy-ribbon knots to all handle-ribbon knots.

The techniques in Section 5 of this paper are very similar to those in [3]. Other sections in this paper rely on the theory of Morse 2-functions.

In this paper, we use twisted Alexander polynomials to prove that four knots (the 3-strand Turk’s head knots with 14, 22, 34, and 46 crossings, respectively) are linearly independent in the knot concordance group, yet all of their prime-powered branched covers bound rational homology balls. None of these knots are of the form (or obviously concordant to a knot of the form) K#K^c, which is the previously known way of constructing such knots (suggested by Chuck Livingston). Conjecturally, there should be an infinitely generated subgroup of such concordance classes in the knot concordance group. Due to the difficulty of computing twisted Alexander polynomials, a strategy for finding an infinite basis of such knots is unclear.

We produce infinitely many genus-one surfaces properly embedded in B^4 that are pairwise topologically isotopic rel. boundary but not smoothly equivalent. The topological isotopy is constructive, although uses Freedman’s theorem that knots with trivial Alexander polynomial are topologically slice. The obstruction to smooth equivalence comes from the knot Floer concordance maps (after suitably decorating the surfaces), which are sensitive to smooth topology. More precisely, we consider maps on perturbed sutured Floer homology.

We extend a concordance version of the 4-dimensional light bulb theorem to the case where one lightbulb has an immersed dual. That is, we prove that if R, R’ are homotopic π₁-negligible (i.e. π₁(R), π₁(R’) include trivially into π₁(X)) surfaces smoothly embedded in a 4-manifold X and G is a framed immersed 2-sphere intersecting R geometrically once, then R and R’ are concordant modulo a condition on 2-torsion in π₁(X). This 2-torsion condition comes from the Freedman-Quinn invariant of a pair of based-homotopic 2-spheres in a 4-manifold, which has recently been studied by Schneiderman and Teichner. We make sense of this invariant for R and R’ freely homotopic in this setting.

We also give explicit counterexamples when G is not framed, using Stong’s Kervaire-Milnor invariant (defined for some pairs of based-homotopic spheres). We believe this to be the first smooth construction of a pair of spheres with nontrivial Stong (Kervaire-Milnor) invariant.
In this paper, I show that there is a finite set $E$ in $S^X$ that is norm-minimizing in $F$. Let $K$ be a knot in $S^3$. We introduce the relative $L$-invariant $rL(X)$ of a smooth, orientable, compact 4-manifold $X$ with boundary. This trisection-theoretic invariant is motivated by the definition of the $L$-invariant for smooth, orientable, closed 4-manifolds by Kirby and Thompson. We show that if $X$ is a rational homology ball, then $rL(X) = 0$ if and only if $X \cong B^4$.

We also prove that any two relative trisections of a given 4-manifold $X$ are related by interior stabilization, relative stabilization, and the relative double twist, which we introduce in this paper as a trisection version of one of Piergallini and Zuddas’s moves on open book decompositions. Previously, it was only known (by Gay and Kirby) that relative trisections inducing equivalent open books on $X$ are related by interior stabilizations. This paper completes the uniqueness classification of relative trisections.

Let $K$ and $J$ be knots in $S^3$. We show that $\nu$-torsion $\text{Ord}_\nu$ in the $\mathcal{F}_2[\nu]$-modules $HF(K), HF^-(J)$ constrain the geometry of cobordisms between $J$ and $K$. In particular, if there is a ribbon concordance from $J$ to $K$ with $n$ local minima, then $\text{Ord}_\nu(K) \leq \max\{n, \text{Ord}_\nu(J)\}$. When $J$ is the unknot, then $K$ is ribbon and this yields $\text{Ord}_\nu(K) \leq \text{Fus}(K)$, the fusion number of $K$. Applying simple algebraic properties of $\text{Ord}_\nu$, we conclude that for any knot $L$, $\text{Ord}_\nu(L) \leq \text{br}(L)$, the bridge-index of $L$.

This is the first result in the literature that relates knot Floer homology to the bridge index of a knot.
Rubinstein and Tillmann constructed multisections of PL n-manifolds, which are decompositions of a PL n-manifold into \( (n+1)/2 \) elementary pieces with well-understood intersections. When \( n = 3 \), a multisection is a Heegaard splitting; when \( n = 4 \), a multisection is a trisection. In this paper, we show how to construct a multisection of a smooth 5-manifold from a handle decomposition. We show that any smooth cobordism between trisectioned 4-manifolds can be multisectioned to agree with the trisections on its boundary.

Although not explicitly written in this paper, the argument also shows how to 4-sect any closed 5-manifold \( X \). That is, how to write \( X = Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \) with the \( Y_i \)’s intersecting only at their boundary and for \( i \neq j \neq k \) we have that \( Y_i \) is a 5D 1-handlebody, \( Y_i \cap Y_j \cap Y_k \) is a 5D 1-handlebody, \( Y_i \cap Y_j \cap Y_k \) is a 3D 1-handlebody, and \( Y_i \cap Y_2 \cap Y_3 \cap Y_4 \) is a closed surface. To see why this is true, just fix a handle decomposition of \( X \) and let \( Y_1 \) be the union of all 0- and 1-handles. Then view \( W := X \setminus Y_1 \) as a cobordism from \( \partial Y_1 \) to \( \emptyset \) and use the results of this paper to obtain a trisection \( (Y_2, Y_3, Y_4) \) of \( W \) with the desired properties.

I consider a concordance analogue of David Gabai’s 4-dimensional light bulb theorem. Gabai’s light bulb theorem says that when \( R \) and \( R' \) are smoothly embedded, homotopic 2-spheres in a 4-manifold \( X \) and admit a common transverse sphere (i.e. a 2-sphere in \( X \) with trivial normal bundle which meets \( R \) and \( R' \) each exactly once, transversely) then \( R \) and \( R' \) are ambiently isotopic, modulo a condition on 2-torsion in \( \pi_1(X) \).

In this paper, I weaken the hypothesis of the light bulb theorem at the cost of weakening the conclusion from ambient isotopy to concordance. More precisely, I show that if \( R \) and \( R' \) are homotopic and \( R \) has a transverse sphere, then \( R \) and \( R' \) are concordant, modulo the same condition on 2-torsion in \( \pi_1(X) \). The proof relies on part of the proof of the 4-dimensional light bulb theorem. Gabai shows that if \( R' \) is in a certain normal form, then \( R' \) is isotopic to \( R \). I show that in the weakened hypotheses, \( R' \) is concordant to a 2-sphere in this normal form.

We show that banded unlink diagrams of a surface \( \Sigma \) in a 4-manifold \( X \) – combinatorial descriptions of the restriction of a Morse function on \( X \) to \( \Sigma \) – are unique up to a finite set of explicit moves. As an application, we show that bridge trisections of surfaces are unique up to a perturbation move, confirming a conjecture of Meier and Zupan. This goes through the original proof of Meier and Zupan, who showed that bridge trisections exist by constructing them from banded unlink diagrams. We also produce several interesting isotopies of spheres in the generating homology class of \( \mathbb{C}P^2 \). We show that for many large families of 2-knots \( K \) known to have Gluck twist diffeomorphic to \( S^4 \), the 2-sphere \( K \# \mathbb{C}P^1 \subset \mathbb{C}P^2 \) is isotopic to \( \mathbb{C}P^1 \). (It holds by work of Melvin that \( (\mathbb{C}P^2, K) \) is diffeomorphic to \( (\mathbb{C}P^2, \mathbb{C}P^1) \).)
I show that if $K$ is a fibered ribbon knot in $S^3$ bounding ribbon disk $D \subset B^4$, then if $D$ satisfies a transversality condition with respect to the fibration on $S^3 \setminus \nu(K)$ then the fibration extends to a fibration of $B^4 \setminus \nu(D)$ by handlebodies. This is related to a question of Casson and Gordon, who showed that $K$ bounds some disk $E$ in a homotopy 4-ball $V$ so that $V \setminus \nu(E)$ is fibered by handlebodies, and so asked whether $V$ can be taken to be $B^4$. When $D$ has two local minima, the transversality condition is automatically satisfied and we conclude that $B^4 \setminus \nu(D)$ is fibered. More generally, I construct a library of twenty simple circular Morse functions with no interior critical points on small 4-manifolds and investigate when these movies can be concatenated to find a circular Morse function with no critical points on a larger 4-manifold (via Cerf theory).

This was my PhD thesis.


We show how to produce a relative trisection description of the complement of any surface smoothly embedded in a smooth 4-manifold. The main tool is a new move on the boundary of a relative trisection. Deleting a bridge-trisected surface from a trisected 4-manifold will in general not yield a relatively trisected manifold, but after performing this boundary move finitely many times, this complement will be relatively trisected. As an application, we show how to produce a trisection description of the result of the Price twist, a certain surgery operation on embedded $\mathbb{R}P^2$'s which can yield exotic 4-manifolds.


I use Gabai’s 4-dimensional light bulb theorem in $S^2 \times S^2$ to construct concordances between homologous surfaces in $S^2 \times S^2$. The existence of these concordances follows from work of Sunukjian. Here we make use of the lightbulb theorem to see the concordances explicitly. This is a short, diagrammatic paper.