Light bulb concordance (20 min)

Thm (4D light bulb) [Gabai 2017]

\[ R, R' \rightarrow X^4 \]

2-spheres which have common dual sphere

Def. Given \( R \times X^4 \), say \( G \) is a dual sphere for \( R \) if \( G \) = 2-sphere w/ triv. normal bundle and \( R \cap G = pt \)

and \( \pi_1(X^4) \) has no 2-torsion.

Then \( R \) and \( R' \) are ambiently isotopic.

If modified statement when \( \pi_1(X^4) \) has 2-torsion: Study homotopy between \( R \) \& \( R' \).

Domain \( S^3 \times I \)

\[ 1 \text{ circle } \Rightarrow \mathbb{S}^2 \]

\[ 2 \text{ circles in gauge } \Rightarrow \mathbb{S}^1 \]

immersed circles of self-int \( \Rightarrow \) element of \( \pi_1(X^4) \)
Gabai's condition: Every element of $T_2 - \{1\}$ appears as a self-intersection of trade of homotopy an even # of times.

$\Rightarrow R, R^1$ isotopic

Thm (Schwartz)

This condition is necessary.

3D analogue

Let $K \subset S^1 \times S^2$ be a circle intersecting $pt \times S^2$ once (transversely). Then $K$ isotopic to $S^1 \times pt$.

(Analogue: $K = R^1, S^1 \times pt = R, pt \times S^2 = G$)
What if \(|K \cap \mathbb{P}^+ \times S^2| > 1\)?

Say \([K] = [S' \times \text{pt}]\) in \(H_1(S' \times S^2)\)

\[ K \not\sim S' \times \text{pt} \quad \text{(Thm: Yildiz, Davis-Nagel-Park-Ray)} \]

But if \([K] = [S' \times \text{pt}]\) in \(H_1(S' \times S^2)\),
then \(K\) concordant to \(S' \times \text{pt}\).
Pf that $K$ concordant to $S^1 \times pt$

- $K$ homotopic to $S^1 \times pt$ through crossing changes

Build concordance from crossing changes

$(S^1 \times S^2) \times 0 \quad (S^1 \times S^2) \times t \quad (S^1 \times S^3) \times 1$

$\rightarrow$

$\rightarrow$
This is a picture of surface

Annulus

no concordance from K to S¹×pt.

Back to dim 'n 4

Then

If $R, R' \subset X^4$ have 2-spheres

$R$ has transverse sphere $G$

and $\pi_1(X^4)$ has no 2-torsion

then $R$ and $R'$ are concordant.

Same condition on homotopy

Feeeman:
5-twist spun trefoil has group $A_5 \times \mathbb{Z}$

surger $\mu$ to get $S^3 > S^3$, K hom to $S^1 \times pt$

$\pi_1(S^3 \times S^3 \setminus K) \cong A_5$, so $K$ not isotopic
to $S^1 \times pt$ (Due to Sato)
4DLBT Proof idea

Step 1

$R'$ can be put into standard form: tubed surface

$\pi_5(\mathbf{i})$  $\pi_5(\mathbf{P})$  $\pi_5(\mathbf{Q})$

single tube
double tube
Regular surfaces

(see e.g. Freedman-Quinn)

Finger move

Whitney move
Thm (Smale)
If $R, R'$ embedded htpc surfaces in $X^n$, then $R$ and $R'$ are reg htpc.

Thm (Quinn?)
Then if a sequence of finger moves $f_1, \ldots, f_n$ followed by Whitney moves $w_1, \ldots, w_n$ (with intermediate isotopies) so $R' \to f_1, f_2, \ldots, f_n, w_1, \ldots, w_n \to R$. 
Have \( R' \) isotopic to tubed surface \( R'' \) on \( R \). 

Gabai:

\[ \rightarrow \text{List} \]

\[ \exists [\gamma] \in \pi_1(X, x) \]

\[ \ni \text{a double tube of } R'' \]

If every 2-torsion element of \( \pi_1(X, x) \) appears an even number of times, then \( R'' \) isotopic to \( R \).
Now prove concordance theorem

Setup $R, R'$ (reg) htpc 2-spheres in $X^4$

$G$ a transverse sphere for $R$

finger moves $f_1, \ldots, f_n$ then

Whitney moves $w_1, \ldots, w_n$

takes $R'$ to $R$. 
Step 1

\[ S = \mathbb{R}^1 \text{ after } f_1, \ldots, f_n \]

\[ S_+ = \text{genus-}n \text{ surface} \]

\[ \text{S immersed but } S_+ \text{ embedded.} \]

Can also obtain \( S_+ \) by attaching tubes to \( \mathbb{R} \).
This gives a cobordism $M$, from $R$ to $S^+$ in $X^4 \times I$ where $M = R \times I$ \cup \text{n 1-handles}$.

To geometrically cancel 1-handles, we want to attach 2-handles along these circles.
Isotope $S_+$ near Whitney moves (in order, tubes disjoint from interior of disk by dimensionality)
Take $B_i$ still centered about point $b_i$ in $R$. Choose arcs from $b_i$ to $z$ and compress $T_i$ along disks parallel to tube around arcs + $G$ to obtain $R''$. 
my cobordism $M_2 \subset X^u \times I$

from $S_+ \to R''$

$M_2 = S_+ \times I \cup \nu \, n \, 2$-handles

$N = M_1 \cup M_2$

= concordance from $R' \to R''$
$\mathbb{R}^n$ is a realization of a tubed surface on $R$. 

Diagram representing a tubed surface.
uncrossed \( w_i \) vs single tubes

crossed \( w_i \) vs double tubes

\( \pi_1 \left( X^4 \right) \) no 2-torsion, then \( R^{11} \) isotopic to \( R \) -> done.

Stop here
Thm (4D LBT Gabai)

Let $R, R'$ be finite 2-spheres embedded in $X^4$ with mutual transverse sphere $G$.

Say finger moves $f_1, \ldots, f_n$ followed by Whitney moves $w_1, \ldots, w_n$ take $R'$ to $R$.

Let $S = R'$ after $f_1, \ldots, f_n$.

Let $(x_1, y_1), \ldots, (x_{2n}, y_{2n})$ be preimages of self-intersections of $S$ (made $2n$ choices).
Say \( w_i \) Whitney move on disk \( W_i \), bandeye \( W_i \) connects

\[
\begin{array}{cc}
\text{uncrossed} & \text{crossed} \\
\end{array}
\]

Let \( \gamma_i \) be the arc in \( X^4 \) with bandeye on \( R \) defining inverse finger move to \( w_i \).

List \( \exists [\gamma_i] \in \pi_1 (X^4, \tau) \) \( w_i \) crossed \( 3 \)

If every element of 2-torsion appears even \# of times, then \( R \) and \( R' \) isotopic.
Thm

Same as above except only $R$ transverse to $G$.

$\implies$ conclude $R$ and $R'$ are concordant.

The remainder of the talk is joint work with Michael Klug.
Let \( R, R' \hookrightarrow X^4 \) 2-spheres w/ common dual \( G \). Assume \( R, R' \) htpc. Then \( R, R' \) isotopic iff \( f_q(R, R') = 0 \).

\( f_q = \) Freedman-Quinn invariant

Defined for homotopic pair of 2-spheres \( A, B \hookrightarrow X^4 \)

\[ f_q(A, B) \subset F_2 T_\mathcal{M} / \mu_3(\pi_3 \mathcal{M}) \]

where \( T_\mathcal{M} \subset \pi_1 X \) is the 2-torsion subset \( (T_\mathcal{M} = \{ 2\text{-torsion elements} \}) \) and \( F_2 T_\mathcal{M} \) is a vector space.
Def \((f_q)\) 

\[ X^4 \times S^2 \times \mathbb{I} \]

\[ C \rightarrow A \]

\[ A \]

Have immersion 

\[ f : S^2 \times \mathbb{I} \rightarrow X^4 \times \mathbb{I} \]

\[ f(S^2 \times 0) = A \times 0 \]

\[ f(S^2 \times 1) = B \times 1 \]

e.g. track of homotopy

Domain

\[ f \]

\[ \Sigma a_i \gamma_i \in \mathbb{F}_2 T_m / M \gamma_3 \gamma_3 M \]

\[ \gamma_i \in T_m, a_i = \# \text{times} \gamma_i \text{ a self-int} \]

(Oriented)

every circle of self-int corresponds to element of

circles of self-intersection
of $f$ with connected double cover (mod 2)

Turns out this does not depend on choice of $f$.

(Up to $\mu_3 \pi_3 M$)

Rmk. If $A, B$ concordant,

then $f_2'(A, B) = 0$.

If $\pi_1 X^4$ no 2-torsion,

then $f_2(A, B) = 0$. 

Sunukjian (2013)

If $R, T \hookrightarrow X$ are 2-spheres

- $R, T$ homotopic
- $i: \pi_1(X \setminus R) \rightarrow \pi_1(X)$ isomorphism (i.e. meridian of $R$ nullhomotopic in $X \setminus R$)
- $\pi_1(X)$ no 2-torsion

then $R$ and $T$ are concordant.

(Didn’t mention this before)
(Because originally no 2-torsion hypothesis)
(Idea of proof)
- First prove for $\pi_1 X = 0$
- Now for other $\pi_1$, lift to universal cover
Problem: What if $y \in \mathbb{Z}$, $X$ fixes some circle setwise? $y^2 = 1$

Thm (Klug - M) Berkeley/Max Planck

Let $R, T \subset X'$ 2-spheres

- $R, T$ homotopic
- $i : \pi_1(X \setminus R) \to \pi_1(X)$ isomorphism
Then $R,T$ concordant iff $f_{(R,T)} = 0$

Lightbulbs (htpc spheres w/ common dual)

- 4-mfds
- 4-mfds
- No 2-torsion
- Gabai
- Schneidersman - Teichner
htpc spheres, one has dual 4-mfds

4-mfds
No 2-torsion
Sunukjian

spheres are concordant

spheres are not concordant

Klug - M

Zeeman:
5-twist spun trefoil $K$ has group $A_5 \times \mathbb{Z}$
surger $\mu$ to get $S^2 \times S^2$, $K$ htpc to $S^2 \times pt$

$\pi_1 (S^2 \times S^2 \setminus K) \approx A_5$ so $K$ not isotopic to $S^2 \times pt$
Redefine $f_q$ in terms of covering spaces

Now $US^2 \times I$ has circle self-ints, which are acted on by $\pi_1 X^4$

If no circle of self-intersection is fixed by some nontrivial $Y \subset \pi_1 X^4$, then Samulejian: Ambiently equivariantly surger the self-intersections away.
Say \( C < M, n M \) a circle of self-int fixed setwise by \( Y \)

so \( Y \cdot 1 \sim 2 \quad Y \cdot 2 = 1 \)

\[ \Rightarrow Y^2 = 1 \]

**Def** \( f_2 (R, T) = \sum a_i Y_i \)

over \( Y_i \in T_2 \) (2-torsion in \( nX \))

\( a_i = \# M_i \cap M \); \( \text{fixed by } Y_i \) (mod 2)

\[ \left( \text{in } \frac{F_2 T_2}{\mu_3 \pi_3 M} \right) \]

**Thm** (KM) Equivalent to

Schneiderman-Teichner definition

\[ \Rightarrow \text{if nonzero, then } R, T \text{ not concordant} \]
And if \( i = 0 \)

( ignore \( \mu_m, \pi_m M \) part)

Then have even \# circles in \( M, nM \) fixed by \( \gamma \)

\[ \simxI \]

equivariantly

\[ \simsurger \]

\( C_1, C_2 \) both fixed set \( \gamma \)

\( C_1, C_2 \) fixed set \( \gamma \)

\( M_1 \)

\( M_2 \)

\[ Z \text{ fixed circles} \]

\[ O \text{ fixed circles} \]
Repeat until no fixed circles, then apply Sunakjian's construction

⇒ R, T are concordant.