Isotopy classification of $\frac{1}{2}$-disks in 4-manifolds,
joint with Danica Kosanović.

A 13 pictures talk based on 2 recent papers.
**Classical LBT:** \( \text{Emb}_\partial (\mathbb{D}^3, \mathbb{M}) \to C^\infty (\mathbb{D}^1, \mathbb{M}) \cong \Omega \mathbb{M} \) induces an isomorphism on \( \pi_0 \), i.e. isotopy \( \Leftrightarrow \) homotopy for knotted arcs \( K \) s.t. \( \exists K' \) has a **dual** \( G : S^2 \to \mathbb{M} \).

**Space-level:** \( \forall d \geq 2, \text{Emb}_\partial (\mathbb{D}^1, \mathbb{M}^d) \cong \Omega (U (M \cup \mathbb{D})) \)

**Thm.** \( \forall d \geq 2, \forall n \leq d \), \( \text{Emb}_\partial (\mathbb{D}^n, \mathbb{M}^d) \cong \Omega (\text{Emb}^\varepsilon_\partial (\mathbb{D}^{n-1}, \mathbb{M}^d)) \) [KT-high-dim, following Cerf (\( n = d \))]

if \( \mathbb{D}^n \) has framed dual sphere \( G : S^{d-n} \to \partial \mathbb{M} \).
\( n = 2, \ d = 3 : \)

**Proof in 2 steps using \( \frac{1}{2} \)-disks \( \mathbb{D}^n := \mathbb{D} \):**

\[
\text{Emb}_{\mathbb{D}}(\mathbb{D}^n, M) \cong \text{Emb}_{\mathbb{D}^n, M_{\mathbb{G}}} \overset{\text{foliate}}{\sim} \Omega \text{Emb}_{\mathbb{D}^n}^\mathbb{E}(\mathbb{D}^n, M_{\mathbb{G}})
\]

\[
M \cong M_{\mathbb{G}} \setminus v(u_+^+) \quad \text{uses only } \text{Emb}_{\mathbb{D}^n}^\mathbb{E}(\mathbb{D}^n, M_{\mathbb{G}}) \cong \nabla
\]

which is the cheap unknotting \( \nabla \).
On each homotopy group $\pi_i$, an inverse of $\Omega \text{Emb}_\mathcal{G}(\mathbb{D}^n, M) \cong \text{Emb}_\mathcal{G}(\mathbb{S}^{n-1}, M)$ is given by the $i$-parameter version of ambient isotopy theorem applied to $U$, e.g. $(n,d,i) = (1,3,0)$, $\gamma : [0,1] \xrightarrow{\text{isotopy}} \text{Emb}_\mathcal{G}([0,\varepsilon), M)$. 

\begin{align*}
\text{Emb}_\mathcal{G}(\mathbb{D}^n, M) &\xrightarrow{\text{foliate}} \Omega \text{Emb}_\mathcal{G}(\mathbb{D}^n, M) \\
\end{align*}
Focus on \((n,d) = (2,4)\) and on \(\Pi_0\), i.e. on isotopy classes.

**Cor.1:**

\[
\mathbb{Z} \left[ \frac{\pi}{1} \right] / \text{dax}(\pi_3 X) \xrightarrow{\text{U+fm(\cdot)}} \text{D}(X; k) \xrightarrow{-\text{U} \cup \text{\cdot}} \pi_2 X
\]

- \(k = \text{U}_- \cup \text{U}_+\) is the \(\frac{1}{2}\) - boundary condition,
- \(\Pi = \Pi_\ast X\), \(X^4\) oriented 4-mfld. with \(\partial X \neq \emptyset\),
- \(\text{U} = "\text{un-}\frac{1}{2}\text{-disk}"\) with boundary \(k\),
- \(\text{Dax} = \text{Dax- invariant}\) for homotopic \(\frac{1}{2}\)-disks.
\[ U \overset{geq \Pi}{\longrightarrow} U + \text{fm}(g) \]

1) Finger move on \( U \) along \( g \in \Pi \),

2) push \( P_t \) off free boundary \( u_t \) along distinct sheets.
Back to neat disks $(\mathcal{D},\mathcal{E}) \hookrightarrow (M^4,\mathcal{E})$ with $\mathcal{D}$-condition $\mathcal{E}$ that has dual $G: S^1 \to M$.

**Cor. 2:** There is a group structure on isotropy classes fitting into a central extension

\[
\mathbb{Z}[\pi] / \text{dax}(\pi_3 M) \xrightarrow{U + \text{fm}(-)^G} \text{Dax} \times eU/2 \xrightarrow{\text{Dax} \times eU/2} \text{D}(M; k) \xrightarrow{pu} \pi_2 M / \mathbb{Z}[\pi] \cdot G
\]

The group commutator of $K_1, K_2$ is

\[
[K_1, K_2] = U + \text{fm}(\lambda(-U \cup K_1, -U \cup K_2))^G.
\]
Here the group \( \mathbb{Z} \left[ \pi_1 \right] / \text{dax}(\pi_3 M) \) acts via

In particular, the group \( D(M; \mathbb{E}) \) is 2-step nilpotent but usually non-abelian, the extension does not split.
We use a subtle extension from $\mathbb{Z}[\pi \backslash 1]$- to $\mathbb{Z}[\pi]$-action by letting $1 \in \pi$ act via

\[ K \longrightarrow K_{tw} \longrightarrow K_{tw}^G : \]
My favorite algebraic topology result is in [KT-4-dim]

**Theorem 3.15.** There is a commutative diagram of short exact sequences of abelian groups for any connected 4-manifold $X$ with $\partial X \neq \emptyset$

\[
\begin{array}{cccccc}
\Gamma(\pi_2 X) & \xrightarrow{\Gamma(- \circ H)} & \pi_3 X & \xrightarrow{\text{Hur}} & H_3 \tilde{X} \\
\downarrow{\Gamma(\mu_2)} & & \downarrow{\text{dax}} & & \downarrow{\mu_3} \\
\mathbb{Z}[\pi \setminus 1] / \langle \bar{g} - g \rangle & \xrightarrow{g \mapsto g + \bar{g}} & \mathbb{Z}[\pi \setminus 1]^\sigma & \xrightarrow{\text{Hur}} & \mathbb{Z}[\pi]^\sigma / \langle 1, g + \bar{g} \rangle & \cong \mathbb{F}_2 [\pi] \\
\end{array}
\]

In particular, $\text{dax}(a \circ H) = \mu_2(a) + \overline{\mu_2(a)} = \lambda(a, a)$ for all $a \in \pi_2 X$,

$$\text{dax} \left( [a_1, a_2]_{\text{Wh}} \right) = \lambda(a_1, a_2) + \lambda(a_2, a_1).$$

and $\mathbb{Z}[\pi \setminus 1]^\sigma / \text{dax}(\pi_3 X)$ is, up to an extension, determined by $\mu_2, \mu_3$. 
