Dax's Work on Embedding Spaces

David Gabai
Princeton University

Topology in Dimension 4.5

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Motivating example

\[ M^3 = S^2 \times D^2 \times S^1 \times B^3 \times [-1, 1] \]

- \( D_0 = \text{std vertical disc} \)
- \( D_1 = \text{self-referential disc} \)
- \( = D_0 \text{ tubed to an unknotted } S^2 \)

Theorem: \( D_1 \) not isotopic to \( D_0 \) rel \( \partial \)

Facts:
1. (LBT) If \( N = M \cup 2\text{-handle} \) along \( \partial D_i \), \( D_i \)'s extended to \( S^3 \)'s then \( S_i \) isotopic to \( S_0 \) fixing \( G \) ptwse.
2. If \( f : M \to M \quad f(D_0) = D_1, \quad f \text{ id fixing } \partial M \text{ ptwise} \) (Based on H. Schwartz spheres using ccf Palais)
3. If \( (D_0, D_1) = 0 \) then \( \pi_1(M) \) is torsion free (See Schneiderman - Teichner)
4. STongs \( K_m (D_0, D_1) = 0 \) \( D_0, D_1 \) have dual spheres (See Klug - Miller)
Theorem \( D_0 \) properly embedded 2-disc \( \subset M^4 \) compact, with dual sphere \( G \subset \partial M \). \( D = \) isotopy classes of embedded discs homotopic to \( D_0 \) rel \( \partial \).

Then there is a homomorphism

\[
\phi_0 : D \to \pi_1^D(Emb(\mathbb{I}, M), \mathbb{I}_0)
\]

\[
= \mathbb{Z}(\pi_1(M)/\mathbb{I})/D(\mathbb{I}_0)
\]

It maps onto subgroup generated by elements \( g+g^{-1} \) and \( x \) where \( x^2 = 1 \).

\( D_0 \) is an abelian group with \([D_0]\) the zero element.

Example when \( M = S^2 \times B^2 \cup S^1 \times B^3 \), \( D(\mathbb{I}_0) = 1 \).

\( D \) is isomorphic to subgroup \( \{z^n + z^{-n}/n \in \mathbb{N}\} \)

\( z \) generator of \( \pi_1 \).
Kosanovic - Teichner

1) Always an $\cong$

2) Understand the space of embeddings $D^2 \rightarrow M^4$
   with boundary dual sphere

3) general group structure
   not necessarily abelian

Hannah Schwartz
LBT for discs with dual sphere $G$ such that
$\pi_i(M-G) \cong \pi_i(M)$.

Question (Schwartz) Is there a
LBT when $\pi_i(M-G) \rightarrow \pi_i(M)$ not $\cong$
Théorème A. — Soient $V^n$ et $M^m$ deux variétés différentielles de classe $C^r$, la variété $V^n$ étant compacte sans bord.

Soit $f : V^n \to M^m$ une application continue. Si $2m - 3n - 3 \geq 0$, $f$ est homotope à un plongement si et seulement si $\pi_0 (f)$ est l'élément neutre du groupe $\Omega_{2n-m} (C_f, \partial W; 0_f)$.

Soient $k$ un entier $\geq 1$ et $f_0 : V^n \to M^m$ un plongement. L'homomorphisme (application pointée si $k = 1$) :

\[ \pi_k : \pi_k (\text{Hom} (V^n, M^m), \text{Pl}, f_0) \to \Omega_{2n-m+k} (C_{f_0}, \partial W; 0_{f_0}) \]

est un isomorphisme (bijection si $k = 1$) pour $k \leq 2m - 3n - 3$, un épimorphisme (surjection si $k = 1$) pour $k = 2m - 3n - 2$.

Jean-Pierre Dax 1972

Étude homotopique des espaces de plongements
Dax Isomorphism Theorem:

\[ \alpha^K_n : \pi^K_n(\text{Hom}(V^K_n, M^K_n), \text{Pl}, f_o) \xrightarrow{\sim} \Omega_{m+k}^D(c_{f_o}, \partial W; \theta_{f_o}) \]

Let \( I_o \) be a properly embedded closed interval in the oriented \( M \).

1) \( \Pi^D_1(\text{Emb}(I, M; I_o)) \) is generated by \( \exists g \mid g \neq 1, g \in \Pi_1(M)^2 \) and is canonically \( \cong \)
\[ \mathbb{Z}[\Pi_1(M) \setminus 1]/D(I_o) \]

2) There is a homomorphism
\[ d_3 : \Pi_3(M, x_0) \longrightarrow \mathbb{Z}[\Pi_1(M) \setminus 1] \]
with image \( D(I_o) \) - the Dax kernel

\( \Pi^D_1(\text{Emb}(I, M; I_o)) \) is the subgroup of loops that are \( \leq \star \) in the space of maps. The \( \text{Dax group} \)

Reference "Self-Referential discs and the light bulb lemma."
**Spinning (Definition of $I_g$)**

- **Convention**
  1. Base of band below lasso on $I_0$
  2. An orientation rule determines sign

**Fact**: Spinnings commute
Dax's key idea:

Let $\alpha_t : I \to M$

with $\alpha_0 = \alpha_1 = 1_{I_0} \circ \text{id}_{I_0}$

$d_t \in \prod^{D}_t(\text{Emb}(I, M), I_0) \Rightarrow$

$\exists \alpha_{t,u} \in \text{Maps}(I, M; I_0)$ with

$\alpha_{t,u} = 1_{I_0} \circ u \text{ near } 0$ \quad $u \text{ near } 1$

Define $F_0 : I \times I^2 \to M \times I^2$

$F_0(s, t, u) = (\alpha_{t,u}(s), t, u)$

we can assume $F$ is an immersion with finitely many double points, no triple points, self at double pts.
Double Points of $F_0$

How to compute $g_x$

Sign $= \sigma_x$

oriented self-intersection $\#$

The orientation of the interval informs which tangent space comes first.
\[ F_0 \rightarrow d(x_{i, u}) = \sum_{i=1}^{n} \sigma_i g_{x_i} \in \mathbb{Z}[\pi_1(M)/1] \]

Summed over double points with \( g_{x_i} \neq 1 \)

This is well defined

If \( x_{i, u}^0, x_{i, u}^1 \) two homotopies in \( \text{Maps}(I, M; I_0) \) and \( x_{i, u}^0 \simeq x_{i, u}^1 \) fix \( \partial \) then the usual intersection theory argument - considering double curves of interpolating homotopy shows \( d(x_{i, u}^0) = d(x_{i, u}^1) \)

Caveat - some double curves come off - but this corresponds to loops \( g_x = 1 \).
If $\alpha_{k,4} \neq \alpha_{t,4}^1$ then they differ by an element of $\Pi_3$.

Define $d_3 : \Pi_3(M, x_0) \to \mathbb{Z}[\Pi_3(M) \setminus \{1\}]$ where $a \in \Pi_3$ is represented by $\alpha_{x_0}^a$ where $\alpha_{x_0}^{x_0}, \alpha_{x_1}^{x_1} = 1_{x_0}$.

Define $D(x_0) = d_3(\Pi_3(M, x_0))$

Then

$d : \Pi_1^0(\text{Emb}(I, M; x_0)) \to \mathbb{Z}[\Pi_1(M) \setminus \{1\}] / D(x_0)$

is a homomorphism.
Looking closely at
\[ F_0 \rightarrow \sum_{i=1}^{n} \sigma_{x_i}, g_{x_i} \in \mathbb{Z}[\pi_1(M)] \]
where the sum is without cancellation and \( g_{x_i} \) possibly 1 then one
see that \( d_\ast \) is a concatenation
of the spin maps \( \sigma_{x_i}, g_{x_i} \).
I.e. \( d_\ast \) differs from \( 1_{I_0} \) by
this concatenation of spinnings
Since spin maps \( \in \Pi_{1}^{D}(\text{Emb}(I, M; I_0)) \)
It follows that
\[ d : \Pi_{1}^{D}(\text{Emb}(I, M; I_0)) \rightarrow \mathbb{Z}[\pi_1(M) \backslash \mathbb{I}] / D(I_0) \]
is surjective. Since spinnings commute and spinning around
\( 1 \) is \( \ast \), \( d \) is injective.
Technical pt: This avoids a double point
s-cobordism like elimination argument
of Dax
Example

\[ \Pi_1^D(\text{Emb}(I, s^1 x B^3; i_0)) \cong \mathbb{Z}[\mathbb{Z} \setminus 0] \]

Proof \[ \Pi_3 = 0 \]

\[ \Pi_1^D(\text{Emb}(I, s^2 x D^2 \sqcup s^1 x B^3; i_0)) \cong \mathbb{Z}[\mathbb{Z} \setminus 0] \]

Proof: Same generators - less space to kill them.
Dax Isomorphism Theorem:

\[ \alpha_k : \pi_k(\text{Hom}(V^h, M^y), \text{Pl}, f_0) \xrightarrow{\cong} \Omega_{\partial W}^{m+k}(e_{f_0}, \partial W; \theta_{f_0}) \]

Let \( I_0 \) be a properly embedded closed interval in the oriented \( M^y \).

i) \( \Pi_i(D(\text{Emb}(I, M; I_0))) \) is generated by \( \forall g \neq 1, g \in \pi_i(M) \) and is canonically \( \cong \) to

\[ \mathbb{Z} [\pi_i(M) \setminus 1] / D(I_0) \]

ii) There is a homomorphism

\[ d_3 : \pi_3(M, x_0) \rightarrow \mathbb{Z} [\pi_1(M) \setminus 1] \]

with image \( D(I_0) \) - the Dax Kernel

Differences between two Dax \( \cong \) Theorems

1) working in different spaces
2) Part ii) is not part of his theory
3) we identify the generators geometrically
\[ \pi^D_1 \left( \operatorname{Emb}(I, S^2 \times D^2 \# S^1 \times B^3; i_0) \right) = \mathbb{Z}[N] \]

**Idea & Proof:** The separating \( S^3 := \Sigma \) gives relations. A 2-sphere \( \Sigma \) bounds \( B^3 \),

One gives spinning \( J_0 \)

Other gives spinning \( J_{g-1} \)

\[ \therefore J_0 \sim J_{g-1} \quad \text{up to sign} \]

\[ \alpha_k : \pi_k^H(\operatorname{Hom}(V, M^k), \Pi, f_0) \rightarrow \Omega_{2n-m+k}^0 (c_{f_0}, \partial W; \theta f_0) \]

These "homotopies" \( \ast, \ast \) represent different elements of the source of \( \alpha_2 \), but are homotopic in \( \pi^D_1 \).
View $D_i \in \Pi_1^D(\text{Emb}(I,M);I_0)$

$d(D_i) = g + g^{-1}$

(using correct choice of sweeping across $D_i$)

**Proof**

$D_i$ is the concatenation of $I_g$, $I_{g^{-1}}$
How to compute $d(D)$ up to $\Sigma(D_0)$ (after H. Schwartz)

Reference (Schwartz) "A LBT for discs"

Step 1: Consider a regular homotopy of $D_0$ to $D$.

Step 2: Consider the self-intersection locus viewed as a plat.

Step 3: Add up crossings (projection into $I, J$ plane) corresponding to identified arcs.

- Each crossing comes with a sign and group element.

Ignore crossings with $g_k = 1$