Topology in Dimension 4.5 – Session C
Recent Results

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Recent Results

Theorem: (Watanabe) $\pi_1 \text{Diff}(D^4)$ is non-trivial.

In this presentation $\text{Diff}(M)$ denotes all diffeomorphisms of $M$ that restrict to the identity on $\partial M$. 
Recent Results

**Theorem:** \((B,G) \pi_0 \text{Diff}(S^1 \times D^3)\) is not finitely-generated.
B,G techniques

Construct essential family $\Omega^{n-j} S^i \to \text{Diff}(B^n_{i,j})$, for any $1 \leq i \leq j < n$.

$B^n_{i,j}$ is the boundary connect-sum of $S^i \times D^{n-i}$ and $S^j \times D^{n-j}$, called a **barbell manifold**.

\[ B^n_{i,j} \equiv S^i \times D^{n-i} \# S^j \times D^{n-j} \]

Notice the first non-trivial homotopy-group of $\Omega^{n-j} S^i$ is in dimension $i + j - n$, and it is $\mathbb{Z}$, provided $i + j \geq n$. 
B,G techniques

Strategy is to find suitable embedding $B^{n}_{i,j} \to N^n$ and extend the family, giving a potentially essential map $\Omega^{n-j}S^i \to \text{Diff}(N)$.

Restriction: The maps

$$\text{Diff}(B^{n}_{i,j}) \to \text{Diff}(S^i \times D^{n-i}) \quad \text{Diff}(B^{n}_{i,j}) \to \text{Diff}(S^j \times D^{n-j})$$

obtained by filling the $j$-handle (or $i$-handle respectively) are null when restricted to our barbell family, thus suitable embeddings $B^{n}_{i,j} \to N$ need to be ‘linked’.
Recent Results - High dimensions

In dimensions $n \geq 6$ we use the ‘Hopf implantation’ of the barbell $B^n_{i,j}$ with $i + j = n$ in $S^1 \times D^{n-1}$.

\[ \pi_{n-j} S^i = \pi_i S^i \equiv \mathbb{Z} \to \pi_0 \text{Diff}(S^1 \times D^{n-1}) \]

These recover* the Hatcher-Wagoner diffeomorphisms of $S^1 \times D^{n-1}$.
Recent Results - High dimensions

In dimension $n = 5$ we use the ‘simple linking’ embedding of $B_{2,2}^5$.

\[ \pi_{5-2}S^2 = \pi_3S^2 \equiv \mathbb{Z} \rightarrow \pi_0\text{Diff}(S^1 \times D^4) \]
Recent Results - High dimensions

In dimension $n = 4$ we use the ‘handcuff embeddings’ of $B_{2,2}^4$. 

$\delta_k$ barbell in $S^1 \times D^{n-1}$
Def’n of barbell diffeomorphism

Consider linearly embedded copies of $\mathbb{R}^i$ and $\mathbb{R}^j$ in $\mathbb{R}^n$, intersecting in a point.

Provided $n - i - j > 0$ we can perturb-away the intersection. Provided $n - i - j > 1$, all such perturbations are isotopic.
Def’n of barbell diffeomorphism – Paint Mixing ODE

Digression: $\Omega^{n-j} S^i \rightarrow \text{Diff}(B_{i,j}^n)$
**Theorem:** (Palais) If $M$ is a manifold, and $N$ a submanifold, the restriction map
\[ \text{Diff}(M) \to \text{Emb}(N, M) \]
is a locally-trivial fibre-bundle.

That the map is a Serre fibration is due to Cerf. The proof is essentially the isotopy-extension theorem ‘with parameters’.

The consequence we need is the homotopy long-exact sequence
\[ \cdots \to \pi_k \text{Diff}(M) \to \pi_k \text{Emb}(N, M) \to \pi_{k-1} \text{Diff}(M \text{ fix } N) \to \cdots \]
specifically the red arrow.
Def’n of barbell diffeomorphism

Consider our map $S^{n-i-j-1} \to \text{Emb}(\mathbb{R}^i \sqcup \mathbb{R}^j, \mathbb{R}^n)$.

The induced element of $\pi_{n-i-j-2} \text{Diff}(\mathbb{R}^n)$ is supported in a ball containing the original double-point.

Given that we can thicken the copies of $\mathbb{R}^i$ and $\mathbb{R}^j$, the diffeomorphism family can be chosen to be the identity on these thickened copies. Thus we have an element of

$$\pi_{n-i-j-2} \text{Diff}(\mathcal{B}^n_{n-i-1,n-j-1})$$

Let $i' = n - i - 1, j' = n - j - 1$ then this is an element of

$$\pi_{i'+j'-n} \text{Diff}(\mathcal{B}^n_{i',j'})$$

This is the barbell diffeomorphism family, on the first non-trivial homotopy group of $\Omega^{n-j'} S^{i'}$. 
Case $i + j = n$, $\pi_i S^i \rightarrow \pi_0 \text{Diff}(B^n_{i,j})$