Banff Topology in Dimension 4.5
Questions and Problems

Patrick Naylor

Q Is the Gluck twist of roll spun $T_3 \cong S^4$?
unnotted $\# = 2$ so this is likely the smallest 2-knot for which we don't know the Gluck twist.

Q For any classical knot $K$, is the turned 1-twisted spun torus of $K$ smoothly unknotted?
Reasonable since $\pi_1(S^1 \times T) \cong \mathbb{Z}$.

What is known about spinning links?
Q Are there similar constructions to deform spins using other ribbon disks?

Anthony Conway

P Find a 2-knot $K \subset S^4$ with trivial Alexander module but nontrivial Rochlin invariant $\mu$.

Q If $K \subset S^4$ has $\pi_1(S^4 \setminus K) \cong \mathbb{Z}$, must $K$ be smoothly unknotted?

This is a well-known question; the affirmative version is called the “smooth unknotting conjecture.”

Q What invariants could help with the previous question? (Also a well-known question.)
Q. Can you define \( \mu \) for nullhomologous 2-knots in other 4-manifolds?

Q. What invariants determine the homotopy type of \( S^4 \setminus \nu(K) \)?

Conjecturally \((\pi_1, \pi_2, k)\) is \"homotopy 2-type\" as a \( \mathbb{Z}_2 \)-module.

This has been studied by e.g. Lomonaco.

Arunima Ray

P. Define \( \tau_g(R, R') \) for more surfaces \( R \) and \( R' \) (e.g. positive-genus, non-orientable).

P. Extend lightbulb theorem (Gabai) to non-orientable ambient 4-manifold.

Q. Can the conditions on the dual sphere \( \Sigma \) in the lightbulb theorem be refined?
Mark Powell

Q Is every 2-link slice?

P Classify $n$-component link maps $\bigsqcup S^2 \to S^4$ up to link homotopy for $n \geq 3$ ($n=1$ trivial; $n=2$ Schneiderman–Teichner)

Rob Schneiderman

Q Are there settings in which $f_q$ can be defined considering unbased homotopies?

E.g., answer is “yes” for $f_q(S^2, S^2)$ when there is an immersed sphere $G \hookrightarrow X^4$ with $G \cdot S^2 \equiv 1 \mod 2$. 
More details: for $H$ a homotopy between 2-spheres in $M^4$, have $f_2(H) \in \mathbb{F}_2 T$ with $T = \mathfrak{g} \in \mathfrak{g}_1 \setminus \mathfrak{g}^2 = 1$, the self-intersection invariant.

$$f_2(H) = \mu(S^2 \times I \to M \times (R \times I)) \in \mathbb{Z}_2 \mathfrak{m} \mathfrak{g}^{-1} \setminus \mathfrak{g}^2 = 1$$

So, for $R, R' \in M$ homotopic 2-spheres, could set

$$f_2(R, R') = f_2(H) \in \mathbb{F}_2 T$$

Problem

If $J$ a basepoint of $R$ traces out a nontrivial element $S \in \pi_1 M$ of $[R] \in \pi_2 M$ under the action of $\pi_1$, get indeterminacy in $f_2(R, R')$ described by $t \mapsto f_2(J) + st \in \mathbb{F}_2 T$ rather than $t \mapsto f_2(J) t$.

- If $f_2(J) \in \pi_2 M$ then on $\mathbb{F}_2 T / (\pi_2 M)$ this reduces to conjugation by $s$.
- If $\text{stab} [R] = 1$ (e.g. $R$ has dual sphere) then $s = 1$.

So, finding a $J$ with $f_2(J) \in \mu(\pi_2 M)$ or finding more examples of $R, M$ where $f_2(J) \in \mu(\pi_2 M)$ & $J$ would help in defining target of $f_2$ without needing a dual sphere.
Q Does there exist a self-homotopy J of some $S^2 \subset M^4$ such that $\mu(J) \neq \mu(\pi_3 M)$? 

\[
(\mu(J) = \sum_{g_c \in \pi_1 M} \begin{array}{c} g_c \in \pi_1 M \\ g \leftrightarrow g^{-1} \end{array})
\]

Answer is "no" if $[S^2] \in \pi_2 M$ has a trivial stabilizer in $\pi_1 M$

e.g. if there is an immersed sphere $G$ with $S^2 \cdot G = 1$.

Ryan Budney

Q Do barbells generate $\pi_0 \text{Diff}(D^4)$?

Or $\pi_0 \text{Diff}(S^1 \times D^3)$?

\[ \Theta_2 : S^1 \times D^3 \text{ see Budney-Gabai} \]

Q Is $\Theta_2$ nontrivial? (Budney-Gabai show $\Theta_n$ nontrivial for $n > 3$.)

Q Can Watanabe's invariants be described in terms of scanning?
Q. Does knotting of barbells matter?

Q. Are the Hatcher-Wagoner invariants surjective in dimension 4?

P. Understand $\text{Diff}(S^2 \times S^2)$ or $\text{Diff}(\mathbb{C}P^2)$. Can you find generators of $\pi_0$?

Q. What is the difference between $\text{Diff}(\text{spin 4-mfld})$ and $\text{Diff}(\text{non-spin 4-mfld})$?

Q. What is $\pi_0 \text{Diff}(D^4)$?
Q Barbells generate the subgroup of \( \pi_0 \text{Diff}(S^1 \times B^{n-1}) \) that is null in pseudoisotopy for \( n \geq 6 \) (Hatcher-Wagoner). Does this hold for \( n = 5 \) too?

P Find null-pseudoisotopies for the Budney-Gabai diffeomorphisms of \( S^1 \times D^3 \) and compute their Hatcher-Wagoner obstructions.

Q Budney-Gabai proved \( \pi_0 \text{Diff}(S^1 \times B^3, s) \) contains an infinite set of linearly independent elements. Are (some of) these elements still non-trivial up to topological isotopy?
Dave Auckly

P. Compare $\pi_n(Diff(Z, D^4))$ to $\pi_n(Homeo(Z, D^4))$ up to stabilizing $Z^4$ by $S^2 \times S^2$.

Tadayuki Watanabe

Q. Do the graph classes in $\pi_k BDiff(D^4)$ survive under the map $\pi_k BDiff(D^4) \to \pi_k BDiff(D^4 \# (S^2 \times S^2))$?

(From the Weiss fiber sequence)

Q. Are the theta-graph (or barbell) classes mapped to nontrivial elements by $\pi_1 BDiff(D^3 \times S^1) \to \pi_1 BHomeo(D^3 \times S^1)$?

- Yes for $\pi_1 BDiff(D^{d+1} \times S^1)$, $d \geq 5$.
- Yes for $\pi_1 BPL(D^3 \times S^1)$. 
Q Can a configuration space integral invariant be defined on \( \pi_1 B\text{Homeo}_2(D^3 \times S^1) \)?

Are invariants of topological embeddings \( \tilde{\Delta}_x \to \tilde{X} \times \tilde{X} \) helpful?

What about in \( \pi_{d-3} B\text{Homeo}_2(D^{d-1} \times S^1) \)?

Q What is the image of

\[ p : \pi_1 M^{\text{psc}}_{2c} (X^4) \to \pi_1 B\text{Diff}_2^c (X^4) \]

moduli space of positive scalar curvature metrics.

- Classes detected by Seiberg-Witten theory are not in \( \text{Im} \ p \) (don't admit fiberwise psc metrics)

- Graph classes are in \( \text{Im} \ p \) (Botvinnik-W 2021)

compare to \( M_{GL}(X^4) \subset \pi_0 \text{Diff}_2^c (X^4) \)

the subgroup for which parameterized Gromov-Lawson construction works; see Botvinnik-Hanke-Schick-Walsh

Gay-Hartman 2022: \( M_{GL}(D^4) \cong \mathbb{Z}/2 \text{ or 0} \).