Deformation inequivalent symplectic structures and Donaldson's four-six question

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Symplectic deformation

Definition

Two closed symplectic manifolds (X_1, ω_1) and (X_2, ω_2) are *deformation equivalent* if there exists a diffeomorphism $\phi : X_1 \to X_2$ such that $\phi^* \omega_2$ is connected to ω_1 via a path of symplectic forms.

- No assumptions on cohomology classes.
- No assumptions on diffeomorphisms.

The "four-six" question

Question (Donaldson)

Given two closed homeomorphic symplectic 4-manifolds (X_1, ω_1) and (X_2, ω_2), are they diffeomorphic if and only if the product manifolds

$$(X_1 \times S^2, \omega_1 \oplus \omega_{\text{std}}) \simeq (X_2 \times S^2, \omega_2 \oplus \omega_{\text{std}})?$$

Call this "stabilization".

Theorem (Wall, '64)

Two closed simply-connected homeomorphic 4-manifolds are h-cobordant.

Theorem (Smale, '62)

Let $n \geq 5$. Then two closed simply-connected n-manifolds are h-cobordant implies that they are diffeomorphic.

History of the question

- Ruan '94: There exist homeomorphic but not diffeomorphic Kähler surfaces such that their stabilizations are not deformation equivalent. This is given by $\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$ and the Barlow surface.
- Ruan-Tian '97: Stated "Stabilizing Conjecture" when restricted to simply-connected 4-manifolds. Shown for rational elliptic surfaces.
- Ionel-Parker '99: Also shown for $E(n)$ using different methods, but still used GW invariants.
- Smith '00: Given $n \geq 2$, constructed n symplectic forms on a simply-connected 4-manifold whose c_1s have different divisibilities. $\Longrightarrow$ Donaldson's question cannot be replaced by $\mathbb{T}^2$.

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Main results

Theorem (Hirschi-W, '23)

There exist infinitely many pairs of closed symplectic 4-manifolds (X_1, ω_1) *and* (X_2, ω_2) *that are diffeomorphic, but their stabilizations* $(X_1 \times S^2, \omega_1 \oplus \omega_{std}) \not\simeq (X_2 \times S^2, \omega_2 \oplus \omega_{std})$.

- In particular, we answer the forward direction of Donaldson's four-six question in the negative.
- We give two "types" of examples, given by Smith and McMullen-Taubes.

Main results

Theorem (Hirschi-W, '23)

The examples above remain deformation inequivalent when stabilized with arbitrarily many copies of (S^2, ω_{std}).

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Proof strategy

- Find examples of a fixed smooth 4-manifold X that admits deformation inequivalent symplectic forms ω_1 and ω_2.
- Show that $(X \times S^2, \omega_1 \oplus \omega_{\text{std}}) \not\simeq (X \times S^2, \omega_2 \oplus \omega_{\text{std}})$.

Invariant: (The orbit under Diff of) the first Chern class associated to a symplectic form.

Goal

Show that $c_1(X \times S^2, \omega_1 \oplus \omega_{\text{std}})$ and $c_1(X \times S^2, \omega_2 \oplus \omega_{\text{std}})$ lie in different orbits of $\text{Diff}(X \times S^2)$.

More motivations for our proof

- Before us, there are already examples of a closed smooth 4-manifold X admitting symplectic strutures whose c_1s lie in different orbits of $\mathrm{Diff}(X)$. In fact, we are using such examples.
- How to go from 4 to 6?
  - Done for diffeomorphisms of $X \times S^2$ that "split".
  - Want to constrain how an arbitrary diffeomorphism of $X \times S^2$ can act on $H^2(X \times S^2)$.

Cohomology equivalences

Definition

Given X and Y, let $G_{X,Y}$ denote the set of cohomology equivalences ψ of $X \times Y$ such that

- ψ^* maps $H^2(X;\mathbb{Z})$ to itself; and
- $\mathrm{pr}_1\psi(\cdot, y)$ is a cohomology equivalence for each $y \in Y$.

Definition

Let $\widetilde{G_{X,Y}} \subset G_{X,Y}$ be the H-group of homotopy equivalences.

☺ Both types of our examples have their diffeomorphisms satisfying one of these algebraic conditions.

Proof steps

- Find a smooth 4-manifold X with symplectic forms ω_1 and ω_2 such that $c_1(\omega_1)$ and $c_2(\omega_2)$ lie in different orbits of ~~$\mathrm{Diff}(X)$~~ cohomology (resp. homotopy) equivalences - this is stronger!
- Show that if $c_1(\omega_1)$ and $c_2(\omega_2)$ lie in different orbits of cohomology (resp. homotopy) equivalences, then $c_1(\omega_1 \oplus \omega_{\text{std}})$ and $c_1(\omega_2 \oplus \omega_{\text{std}})$ lie in different orbits of G_{X,S^2} (resp. $\widetilde{G_{X,S^2}}$).
- Show that any diffeomorphism of $X \times S^2$ lies in G_{X,S^2} (resp. $\widetilde{G_{X,S^2}}$).

Algebraic "sufficient" condition

Proposition

Let X be a closed, smooth manifold with two symplectic forms ω_1 and ω_2. Suppose $c_1(\omega_1)$ and $c_1(\omega_2)$ are in different orbits of actions of cohomology (resp. homotopy) equivalences of X on $H^2(X;\mathbb{Z})$. Then $c_1(\omega_1 \oplus \omega_{\text{std}})$ and $c_1(\omega_2 \oplus \omega_{\text{std}})$ lie in different orbits of action of G_{X,S^2} (resp. $\widetilde{G_{X,S^2}}$).

Proof of the algebraic "sufficient" condition

Suppose $\exists \psi \in G_{X,S^2}$ (resp. $\widetilde{G_{X,S^2}}$), such that

$$\psi^* c_1(\omega_2 \oplus \omega_{\text{std}}) = c_1(\omega_1 \oplus \omega_{\text{std}}). \tag{1}$$

Then for $h := \mathrm{PD}[\mathrm{pt}] = \mathrm{AD}[S^2] = c_1$ of the hyperplane line bundle, we have that $\psi^* h = h + \alpha$ for some $\alpha \in H^2(X)$. Also, $\psi^*(h^2) = 0$, implying

$$(h + \alpha)^2 = h^2 + 2\alpha h + \alpha^2 = 0.$$

So $2\alpha = 0$, since $H^*(X \times S^2; \mathbb{Z}) \cong H^*(X)[h]/h^2$. Now, by (1) and $c_1(\mathbb{C}P^1, \omega_{\text{std}}) = 2h$, we have that

$$c_1(\omega_1) + 2h = \psi^* c_1(\omega_2) + 2\psi^* h = \psi^* c_1(\omega_2) + 2h + 2\alpha.$$

So $c_1(\omega_1) = \psi^* c_1(\omega_2)$. Let $\widehat{\psi} := \mathrm{pr}_1(\psi(\cdot, z))$ for some $z \in S^2$. Since ψ^* preserves $H^2(X; \mathbb{Z})$,

$$\widehat{\psi}^* c_1(\omega_2) = \psi^* c_1(\omega_2) = c_1(\omega_1).$$

Since $\widehat{\psi}$ is a cohomology (resp. homotopy) equivalence, contradiction.

Counterexamples

Example (Smith, 2000)

Let $Z := (\mathbb{T}^4_{(x,y,z,t)}, dxdt + dydz)\#_{\text{fiber sum}} 5(E(1), \omega_0)$ along $T_x := \langle x, t \rangle$, $T_y := \langle y, t \rangle$, $T_z := \langle z, t \rangle$ and 2 copies of $T_w := \langle x = y = z, t \rangle$.
Can check that T_x, T_w are symplectic, while T_y, T_z are Lagrangian.

Theorem (Smith, 2000)

The simply-connected manifold Z admits two deformation inequivalent symplectic forms ω_+ and ω_- obtained by perturbing T_z "via the opposite orientation". In fact, $3|c_1(\omega_+)$ but not $c_1(\omega_-)$.

Counterexamples

Theorem (Hirschi-W, '23)

Let Z be the simply-connected 4-manifold constructed by Smith. Then $\omega_+ \oplus \omega_{std}$ and $\omega_- \oplus \omega_{std}$ on $Z \times S^2$ are deformation inequivalent.

Proof idea: uses the fact that $p_1(Z \times S^2) = p_1(Z) = m\mathrm{PD}([S^2])$ for some $m \neq 0$ and that p_1 is preserved by diffeomorphisms up to sign to show that any diffeomorphism of $Z \times S^2$ must lie in G_{X,S^2}.

Counterexamples

Example (McMullen-Taubes, 1999)

Let $L := L_1 \sqcup L_2 \sqcup L_3 \sqcup L_4$ consist of four closed geodesics representing the three S^1-factors of $\mathbb{T}^3$ and a fourth component satisfying $[L_4] = [L_1] + [L_2] + [L_3]$.
Consider $M := \mathbb{T}^3 \setminus \mathcal{N}(L)$. Let N be the double branched cover of $\mathbb{T}^3$ over L associated to the homomorphism

$$\xi : H_1(M; \mathbb{Z}) \to \{\pm 1\}$$

with $\xi(m_4) = -1$, where m_4 is the meridian of L_4.

McMullen-Taubes example continued

Then N is fibered, induced from a fibration $\mathbb{T}^3 \to S^1$.

Definition

The Euler class of a fibration $\rho : N \to S^1$ only depends on $\alpha = [d\rho] \in H^1(N;\mathbb{Z})$ and is given by

$$e(\alpha) = [s^{-1}(0)] \in H_1(N;\mathbb{Z})/\text{torsion},$$

where $s : N \to \ker(d\rho)$ is a generic section.

McMullen-Taubes example continued

Lemma (McMullen-Taubes, 1999)

3D *There exist* $\alpha_1, \alpha_2 \in H^1(N; \mathbb{Z})$ *induced by fibrations* $\rho_1, \rho_2 : N \to S^1$ *such that* $e(\alpha_1)$ *and* $e(\alpha_2)$ *lie in different orbits of* $Aut(\pi_1(N))$*-action on* $H_1(N; \mathbb{Z})$.

4D *Furthermore, let* $X := N \times S^1$. *One can associate an* S^1*-invariant form* ω *to each* $\alpha \in H^1(N; \mathbb{Z})$ *represented by the differential of a fibration and*

$$c_1(X, \omega) = PD_X(e(\alpha) \times [S^1]).$$

Corollary

For any homotopy equivalence ϕ *of* X, *we have that* $\phi_*(e(\alpha_1) \times [S^1]) \neq e(\alpha_2) \times [S^1]$, *implying* $\phi^* c_1(\omega_2) \neq c_1(\omega_1)$.

McMullen-Taubes example continued

Remains to show that any diffeomorphism of $X \times S^2$ lies in $\widetilde{G_{X,S^2}}$. This essentially follows from the fact that X is aspherical. Therefore, the McMullen-Taubes construction also gives counterexamples to Donaldson's $4-6$ question.

Thank you for listening!