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ORTHOGONAL SYMMETRIC CHAIN DECOMPOSITIONS OF HYPERCUBES*

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Abstract. In 1979, Shearer and Kleitman conjectured the existence of $\lfloor n/2 \rfloor + 1$ orthogonal chain decompositions of the hypercube poset Q_n and constructed two orthogonal chain decompositions. In this paper, we make the first nontrivial progress on this conjecture by constructing three orthogonal chain decompositions of Q_n for all $n \geq 4$ with the possible exceptions $n = 9, 11, 13, 23$. To do this, we introduce the notion of "almost orthogonal symmetric chain decompositions." We explicitly describe three such decompositions of Q_5 and Q_7 and describe conditions which allow us to decompose products of hypercube posets into k almost orthogonal symmetric chain decompositions given such decompositions of the original hypercube posets.

Key words. set systems, chain decomposition, symmetric chain decomposition, hypercube, orthogonal

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1. Introduction.

In this paper we will consider chain decompositions of the hypercube poset Q_n, which may be viewed either as the poset of subsets of $[n] := \{1, \ldots, n\}$ under inclusion or as $\{0, 1\}^n$ under the ordering $(a_1, \ldots, a_n) \preceq (b_1, \ldots, b_n)$ if $a_i \leq b_i$ for every $1 \leq i \leq n$. One of the first major applications of chain decompositions of Q_n was in 1970 by Kleitman [7], resolving the Littlewood–Offord problem on sums of Bernoulli random variables. The key step was an inductive decomposition of Q_n into chains, each of which contains for some k precisely one element of each of the ranks $k, k+1, \ldots, n-k$; such chains are called *symmetric*, and such a decomposition is called a *symmetric chain decomposition*. Note that every symmetric chain contains an element of size $\lfloor n/2 \rfloor$, so a symmetric chain decomposition always has $\binom{n}{\lfloor n/2 \rfloor}$ elements (the smallest possible for any chain decomposition). This inductive decomposition was later streamlined by Greene and Kleitman [3] using a clever parenthesis-matching argument, which has recently seen surprising applications in the theory of crystal bases (see, e.g., [5], where rederiving a result of Jordan [6], Hersch and Schilling use Lyndon words on parentheses arising from crystal bases to decompose $Q_n/(\mathbb{Z}/n\mathbb{Z})$ into symmetric chains).

An interesting application of the Greene–Kleitman parenthesis-matching argument was in 1979 by Shearer and Kleitman [10] to find a lower bound on the probability that two subsets of $\{1, \ldots, n\}$ are comparable as we vary over all probability distributions on Q_n. To accomplish this, they constructed two "orthogonal chain decompositions" of Q_n by a slight modification of two decompositions found by the parenthesis-matching argument. Two decompositions $\mathcal{F}$ and $\mathcal{G}$ of Q_n into $\binom{n}{\lfloor n/2 \rfloor}$ chains are said to be *orthogonal chain decompositions* [10] if for any chain $A \in \mathcal{F}$ and $B \in \mathcal{G}$, we have $|A \cap B| \leq 1$. It was conjectured in [10] that $\lfloor n/2 \rfloor + 1$ orthogonal chain decompositions of Q_n exist, i.e., there exist decompositions $\mathcal{F}^1, \ldots, \mathcal{F}^{\lfloor n/2 \rfloor + 1}$ of Q_n,

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each into $\binom{n}{\lfloor n/2 \rfloor}$ chains, such that $\mathcal{F}^j$ and $\mathcal{F}^{j'}$ are orthogonal chain decompositions for every $j \neq j'$.

We note that a somewhat weaker notion of edge-disjoint decompositions, first considered by Pikhurko [9], was recently used by Mütze [8] to prove the middle levels conjecture, and Gregor et al. [4] have created four edge-disjoint symmetric chain decompositions of Q_n for $n \geq 12$ to extend this result to the middle four levels.

Our main result, making the first progress on the conjecture of Shearer and Kleitman, is that Q_n has three orthogonal chain decompositions for $n \geq 4$ with the possible exceptions of $n = 9, 11, 13, 23$. We accomplish this by carefully constructing certain decompositions of *cuboid posets* (i.e., posets formed by the product of chain posets) to bootstrap orthogonal chain decompositions of small dimensional hypercubes to arbitrary products of such hypercubes.

There are serious technical obstructions to accomplishing this, and we must develop a robust theory of products of what we call "almost orthogonal symmetric chain decompositions." This theory doesn't use the poset structure of Q_n in an essential way (we only use information about lengths of symmetric chains), and so similar considerations apply to products of other posets. We leave this as a direction for future research.

Remark 1.1. Using Theorem 3.5 concerning bootstrapping "almost orthogonal symmetric chain decompositions," Däubel et al. [1] have recently produced four orthogonal chain decompositions of Q_n for every $n \geq 60$ by finding four "almost orthogonal symmetric chain decompositions" of Q_7 and Q_{11} via computer search.

2. Definitions. We start by recalling the following definitions from [10].

DEFINITION 2.1 (see [10]). *Two chains $A, B \subset Q_n$ are* orthogonal *if $|A \cap B| \leq 1$. Two decompositions $\mathcal{F}$ and $\mathcal{G}$ of Q_n into $\binom{n}{\lfloor n/2 \rfloor}$ chains are orthogonal if for any chains $A \in \mathcal{F}$ and $B \in \mathcal{G}$, we have A and B orthogonal. A family of decompositions $\{\mathcal{F}^1, \ldots, \mathcal{F}^k\}$ of Q_n into $\binom{n}{\lfloor n/2 \rfloor}$ chains is orthogonal if $\mathcal{F}^j$ is orthogonal to $\mathcal{F}^{j'}$ for every $j \neq j'$.*

Recall a *symmetric chain decomposition* of Q_n is a decomposition of Q_n into *symmetric chains*, i.e., for each chain there is an integer k such that the chain consists of one element of size $k, k+1, \ldots, n-k$; this is always a decomposition into $\binom{n}{\lfloor n/2 \rfloor}$ chains since each chain contains a unique $\lfloor n/2 \rfloor$-element subset. A symmetric chain is an example of a *skipless chain*, a chain whose elements have ranks which form an interval in $\mathbb{Z}$. The following definition is at the heart of this paper.

DEFINITION 2.2. *We call two chains $A, B \subset Q_n$* almost orthogonal *if either A, B are orthogonal or A, B are maximal chains with $A \cap B = \{\emptyset, [n]\}$. We say that symmetric chain decompositions $\mathcal{F}$ and $\mathcal{G}$ of Q_n are almost orthogonal if for any chains $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are almost orthogonal. We say a family of symmetric chain decompositions $\{\mathcal{F}^1, \ldots, \mathcal{F}^k\}$ of Q_n is almost orthogonal if $\mathcal{F}^j$ is almost orthogonal to $\mathcal{F}^{j'}$ for every $j \neq j'$.*

To formulate Theorems 3.5 and 3.6, we state the following technical condition, which will be fully motivated in section 7.

DEFINITION 2.3. *We say that a collection of almost orthogonal symmetric chain decompositions $\{\mathcal{F}^j\}_{1 \leq j \leq k}$ of Q_n is good if either n is even and the one-element chains in all $\mathcal{F}^j$ are distinct or n is odd and the graph with vertex set the union of all* *two-element chains in all* $\mathcal{F}^j$ *and edges the two-element chains themselves can have its edges oriented so that every vertex has out-degree at most* 1 *(equivalently, every component is a tree or a tree union an edge).*

Finally, as we will often have to work with the product of chain posets, we make the following definitions.

DEFINITION 2.4. *Given a chain poset* $\mathcal{C}_i$*, we define its* length *to be the number of elements.*

We define a cuboid of dimension $d_1 \times \cdots \times d_r$ *to be the product poset* $\prod_{i=1}^r \mathcal{C}_i$*, where* $\mathcal{C}_i$ *is a chain poset of length* d_i *and where by definition* $(a_1, \ldots, a_r) \preceq (b_1, \ldots, b_r)$ *if* $a_i \preceq b_i$ *for* $1 \leq i \leq r$*. We define a* rectangle *to be a cuboid with* $r = 2$*.*

3. Main results. We now state our main results. Our proofs of Theorem 3.3, Corollary 3.4, and Theorem 3.5 appear in section 10 and our proof of Theorem 3.6 appears in section 11, though the auxilliary results required for our proofs span the entire paper.

The following proposition is proved in subsection 5.3.

PROPOSITION 3.1. *Suppose that* Q_n *has* $k \geq 3$ *almost orthogonal symmetric chain decompositions* $\mathcal{F}^1, \ldots, \mathcal{F}^k$ *for some* $n \geq 5$*. Then* Q_n *has* k *orthogonal chain decompositions.*

Remark 3.2. One can exhaustively check that Q_4 does not have three almost orthogonal symmetric chain decompositions, but we will not need or use this fact.

We now state our main result on almost orthogonal symmetric chain decompositions.

THEOREM 3.3. *There exist three almost orthogonal symmetric chain decompositions of* Q_n *for all* $n \geq 5$ *except possibly* $n = 6, 8, 9, 11, 13, 16, 18, 23$*.*

The exceptions are precisely those n not representable as a nonnegative integral combination of 5 and 7. From this, we are able to show the following, which is the main result of this paper.

COROLLARY 3.4. *There exist three orthogonal chain decompositions of* Q_n *for all* $n \geq 4$ *except possibly* $n = 9, 11, 13, 23$*.*

This makes progress on the conjecture of Shearer and Kleitman [10] (which was verified up to $n = 4$) that there exist $\lfloor n/2 \rfloor + 1$ orthogonal chain decompositions of Q_n, easily shown to be an upper bound by considering elements of middle rank(s). Of independent interest is the following bootstrapping result for $k \geq 3$ almost orthogonal symmetric chain decompositions.

THEOREM 3.5. *For* $1 \leq i \leq r$*, suppose we have* $k \geq 3$ *almost orthogonal symmetric chain decompositions* $\{\mathcal{F}_i^j\}_{1 \leq j \leq k}$ *of* Q_{n_i} *with* $n_i \geq 4$*. Suppose the collections associated to* Q_{n_i} *with* n_i *even are good and either all or at least two of the collections associated to the* Q_{n_i} *with* n_i *odd are good. Then* $Q_{n_1+\ldots+n_r}$ *has* k *almost orthogonal symmetric chain decompositions.*

We will see good families of decompositions in section 4 for $k = 3$ and $n = 5, 7$, which combined with Theorem 3.5 proves Theorem 3.3. Using a very specific symmetric chain decomposition provided in [2] of cuboids formed as the product of at least five two-element chains with a chain of length ≥ 5, we can remove the goodness hypothesis in Theorem 3.3 completely for the odd dimensional hypercubes provided there are at least six of them appearing in the product. As we will see, cuboids with all but one side of length 2 prove to be some of the most challenging ones to handle.

THEOREM 3.6. *For $1 \leq i \leq r$, suppose we have $k \geq 3$ almost orthogonal symmetric chain decompositions $\{\mathcal{F}_i^j\}_{1\leq j\leq k}$ of Q_{n_i} with $n_i \geq 4$. Suppose the collections associated to Q_{n_i} with n_i even are good, and at least six of the n_i are odd. Then $Q_{n_1+\ldots+n_r}$ has k almost orthogonal symmetric chain decompositions.*

Finally, there is one more theorem, Theorem 8.1, whose statement requires the definitions from section 6, which is the cornerstone of our paradigm for decomposing products of almost orthogonal symmetric chain decompositions, and should allow for generalization to products of posets other than hypercube posets. We further elaborate on this in section 12.

4. Three good almost orthogonal symmetric chain decompositions of Q_5 and Q_7. In this section we describe three almost orthogonal symmetric chain decompositions of Q_5 and Q_7. We will also verify that these decompositions are good in the sense of Definition 2.3.

4.1. Three almost orthogonal symmetric chain decompositions of Q_5. In this subsection we produce three almost orthogonal symmetric chain decompositions of Q_5 which are good. To compactify notation, we omit the empty set and the whole set from our chains, and we further suppose that if a chain is written, then all chains found by cycling the indices should also be written. An easy check shows that these decompositions are almost orthogonal to each other. The 15 chains of length 2 form five paths with three edges each, so this verifies that the family of decompositions is good.

$\mathcal{F}_1$:	1	12	123	1235	∅ and 12345 added onto the chain starting with 1
		14	134		
$\mathcal{F}_2$:	1	15	135	1345	∅ and 12345 added onto the chain starting with 5
		14	145		
$\mathcal{F}_3$:	1	14	124	1234	∅ and 12345 added onto the chain starting with 3
		12	125		

4.2. Three almost orthogonal symmetric chain decompositions of Q_7. We now produce three almost orthogonal symmetric chain decompositions of Q_7. Grouping the elements modulo the equivalence relation given by cycling the indices, one can readily check the finite number of cases to show that the decompositions below are almost orthogonal to each other. The two-element chains form seven paths with two edges each, seven isolated edges, and seven star graphs formed by three edges that share a common vertex, so this verifies the family of decompositions is good.

$\mathcal{F}_1$:	1	12	123	1234	12345	123456	∅ and 1234567 added onto chain starting with 1
		13	134	1345	13456		
		14	147	1247	12457		
			124	1245			
			135	1357			
$\mathcal{F}_2$:	1	17	167	1567	14567	134567	∅ and 1234567 added onto chain starting with 7
		13	137	1357	13567		
		14	145	1345	13457		
			134	1347			
			135	1235			
$\mathcal{F}_3$:	1	13	136	1346	13467	123467	∅ and 1234567 added onto chain starting with 6
		12	124	1247	12347		
		14	134	1234	12346		
			123	1236			
			125	1245			

5. Product constructions and our general strategy. In this section we outline what is known about product constructions of symmetric chain decompositions, what the obstructions to generalizing Shearer and Kleitman's construction of two almost orthogonal symmetric chain decompositions to $k \geq 3$ decompositions are, and what our general strategy for the remainder of this paper is. We also prove Proposition 3.1, which yields k orthogonal chain decompositions from k almost orthogonal symmetric chain decompositions.

5.1. Product decompositions for symmetric chain decompositions. Both the inductive decomposition of Kleitman and the parenthesis-matching argument of Greene and Kleitman are special cases of the following classical observation in the special case $Q_{n+1} \cong Q_n \times Q_1$.

Let $n_i \geq 1$ be integers for $1 \leq i \leq r$, and let $n = \sum_{i=1}^r n_i$. Given symmetric chain decompositions $\mathcal{F}_i$ of the hypercubes Q_{n_i}, we can produce a symmetric chain decomposition of $Q_n \cong \prod_{i=1}^r Q_{n_i}$ by decomposing for each choice of symmetric chains $\mathcal{C}(\mathcal{F}_i) \in \mathcal{F}_i$ the product cuboid $\prod_{i=1}^r \mathcal{C}(\mathcal{F}_i) \subset \prod_{i=1}^r Q_{n_i} \cong Q_n$ into symmetric chains. Note that we have a large amount of freedom in how we do this, and we can in fact repeatedly apply the case $r = 2$ instead of directly decomposing high dimensional cuboids, where one can directly produce symmetric chain decompositions for any dimensions of rectangles.

Applying this to $Q_{n+1} \cong Q_n \times Q_1$ yields the following symmetric chain decomposition of Q_{n+1} from a symmetric chain decomposition of Q_n. For every chain $A : a_k \subset a_{k+1} \subset \cdots \subset a_{n-k}$ in the decomposition of Q_n, let A' be the chain $a_k \cup \{n+1\} \subset \cdots \subset a_{n-k} \cup \{n+1\}$. Neither chain is symmetric in Q_{n+1}: A is one element too low, and A' is one element too high. If we move the largest element from A' to A or the smallest element from A to A', however, we now have two symmetric chains (if one of the chains happens to be empty after doing this, discard it). Repeating this for every chain in the decomposition of Q_n, we get a decomposition of Q_{n+1} into symmetric chains.

Remark 5.1. The two possibilities for modifying A and A' correspond to the two possible symmetric chain decompositions of a $2 \times k$ rectangle (see, for example, Figure 2).

5.2. Two (almost) orthogonal chain decompositions of $\boldsymbol{Q_n}$. Suppose we have two almost orthogonal symmetric chain decompositions $\mathcal{F}, \mathcal{G}$ of Q_n. We may rephrase the argument of Shearer and Kleitman [10] as producing two almost orthogonal symmetric chain decompositions of Q_{n+1} as follows.

For every chain $A \in \mathcal{F}$, we apply the procedure from the previous subsection moving the largest element of A' to A to produce our first symmetric chain decomposition of Q_{n+1}, and for every chain $B \in \mathcal{G}$ apply the procedure from the previous subsection moving instead the smallest element of B to B' to produce our second symmetric chain decomposition of Q_{n+1}. It is an easy check that this yields almost orthogonal symmetric chain decompositions of Q_{n+1}. We needed to change how we moved the elements, because if we had, for example, moved the smallest element of A to A' and the smallest element of B to B', and A, B shared the same smallest element, then A' and B' would have shared the same smallest two elements, contradicting orthogonality.

To produce orthogonal chain decompositions from the almost orthogonal decompositions, it suffices to move $\emptyset$ from the maximal chain in $\mathcal{F}$ to one of the minimal length chains in $\mathcal{F}$ disjoint from the maximal chain in $\mathcal{G}$.

Remark 5.2. We can see what the problem is in propagating larger families of almost orthogonal symmetric chain decompositions to higher dimensional hypercubes—it might be impossible to choose the method of duplicating the chains in such a way that it preserves orthogonality, since we now possibly have ≥ 3 chains meeting at a point instead of two, forbidding us from simply "doing the opposite" to each of the families. In fact, it is already impossible if we just consider the maximal chains in each decomposition.

5.3. Almost orthogonal symmetric chain decompositions to orthogonal decompositions. We now prove Proposition 3.1 by showing more generally that almost orthogonal symmetric chain decompositions induce orthogonal chain decompostions by moving around $\emptyset$.

Proof of Proposition 3.1. We in fact show that we can move $\emptyset$ from the maximal chain in each $\mathcal{F}^j$ to nonintersecting minimal length chains, which produces k orthogonal chain decompositions.

Note that by considering chains of middle rank(s) we have that $k \leq \lfloor n/2 \rfloor + 1$. The number of minimal length chains in Q_n is

$$\binom{n}{\lfloor n/2 \rfloor} - \binom{n}{\lfloor n/2 \rfloor - 1},$$

and from the inequality on k and the fact that $n \geq 5$, this quantity is strictly larger than $k-1$ if n is even and $2(k-1)$ if n is odd. As the minimal length chains in Q_n are of length 1 if n is even and 2 if n is odd, this shows we may move $\emptyset$ to a chain of minimal length one $\mathcal{F}^j$ at a time, and some minimal length chain will always be available at each step. □

5.4. Products of $k \geq 3$ almost orthogonal symmetric chain decompositions of Q_m, Q_n. The problem addressed in Remark 5.2 demonstrates we cannot bootstrap k almost orthogonal symmetric chain decompositions of Q_n to k almost orthogonal symmetric chain decompositions of Q_{n+1} when $k \geq 3$. However, we will show in section 7 that given k almost orthogonal symmetric chain decompositions $\mathcal{F}^1, \dots, \mathcal{F}^k$ of Q_m and $\mathcal{G}^1, \dots, \mathcal{G}^k$ of Q_n with $m, n \geq 2$ such that $\{\mathcal{F}^j\}_{1 \leq j \leq k}$ and $\{\mathcal{G}^j\}_{1 \leq j \leq k}$ are good, we will be able to decompose the product rectangles in each $\mathcal{F}^j \times \mathcal{G}^j$ for $1 \leq j \leq k$ in a consistent way so that the resulting k symmetric chain decompositions of Q_{m+n} are orthogonal symmetric chain decompositions. In other words, we can take two collections of good almost orthogonal decompositions and produce an orthogonal decomposition of the product hypercube.

Remark 5.3. Because we lose the goodness of the collection of decompositions on the product hypercube, we cannot simply iterate this result to conclude for products of arbitrarily many hypercubes.

5.5. Products of $k \geq 3$ almost orthogonal symmetric chain decompositions of $Q_{n_1}, \dots, Q_{n_r}$. Finally, to overcome the obstacle that our notion of "goodness" isn't preserved in product constructions, we will not apply the inductive simplification of working with decompositions of rectangles. Instead, we will work with decompositions of high dimensional cuboids directly.

Hence, the situation we are in is as follows. We have a collection of k almost orthogonal symmetric chain decompositions $\mathcal{F}_i^1, \dots, \mathcal{F}_i^k$ of Q_{n_i} for each $1 \le i \le r$, and some collections are good according to the hypotheses of either Theorem 3.5 or Theorem 3.6. Our goal is to produce k almost orthogonal symmetric chain decompositions of $\prod_{i=1}^r Q_{n_i} \cong Q_n$. To do this, we take for each $1 \le j \le r$ the product cuboids from $\prod_{i=1}^r \mathcal{F}_i^j$ and decompose them into symmetric chains in such a way that symmetric chains from a product cuboid associated to j and to j' with $j \ne j'$ intersect almost orthogonally. The result will be a collection of k almost orthogonal symmetric chain decompositions of Q_n.

6. Proper and very proper decompositions. In this section, we introduce the key technical tool for creating almost orthogonal symmetric chain decompositions via a product construction. For the remainder of this section, we will consider hypercubes $Q_{n_1}, \dots, Q_{n_r}$ such that for $1 \le i \le r$, Q_{n_i} has a pair of almost orthogonal symmetric chain decompositions $\{\mathcal{S}_i, \mathcal{T}_i\}$. We will denote by $n = n_1 + \cdots + n_r$, and we will denote by $\mathcal{C}(\mathcal{S}_i)$ and $\mathcal{C}(\mathcal{T}_i)$ some choices of chains in $\mathcal{S}_i$ and $\mathcal{T}_i$ respectively for each i.

6.1. Proper and very proper chains. Recall that to produce a symmetric chain decomposition of $Q_n \cong Q_{n_1} \times \cdots \times Q_{n_r}$ from the $\mathcal{S}_i$ decompositions, we are tasked to decompose the cuboid $\prod_{i=1}^r \mathcal{C}(\mathcal{S}_i)$ into symmetric chains for each choice of chains $\mathcal{C}(\mathcal{S}_i) \in \mathcal{S}_i$.

Our key definition of "properness" will be a property of a skipless chain

$$\mathcal{C} \subset Q_n \cong Q_{n_1} \times \cdots \times Q_{n_r}$$

so that if $\mathcal{C} \subset \prod_{i=1}^r \mathcal{C}(\mathcal{S}_i)$ is "proper," then the intersection of $\mathcal{C}$ with every symmetric chain in $\prod_{i=1}^r \mathcal{C}(\mathcal{T}_i)$ is almost orthogonal. The definition will depend only on the decomposition $Q_n \cong Q_{n_1} \times \cdots \times Q_{n_r}$ and will be more stringent the finer the decomposition is.

DEFINITION 6.1. *Given a skipless chain $\mathcal{C} \subset Q_n \cong Q_{n_1} \times \cdots \times Q_{n_r}$, we say that $\mathcal{C}$ is* proper *with respect to the decomposition $Q_n \cong Q_{n_1} \times \cdots \times Q_{n_r}$ if for every pair of distinct elements $x = (x_1, \dots, x_r), y = (y_1, \dots, y_r) \in \mathcal{C}$, either*

- *$\{x, y\} = \{\emptyset, [n]\}$ or*
- *there is a coordinate i such that $x_i \ne y_i$ and $\{x_i, y_i\} \ne \{\emptyset, [n_i]\}$.*

Say that a proper chain is very proper *if it is nonmaximal in Q_n. Given skipless chains $\mathcal{C}_i \subset Q_{n_i}$, say a symmetric chain decomposition of $\mathcal{C}_1 \times \cdots \times \mathcal{C}_r$ is (very) proper if every chain is (very) proper in $Q_n \cong Q_{n_1} \times \cdots \times Q_{n_r}$.*

Example 6.2. If $r = 1$, then every skipless chain is proper, and every non-maximal skipless chain is very proper. In $Q_2 \times Q_2$, $(\emptyset, \emptyset) - (\emptyset, \{1\}) - (\{1\}, \{1\}) - ([2], \{1\}) - ([2], [2])$ is not proper because of $(\emptyset, \{1\})$ and $([2], \{1\})$. However, $(\emptyset, \emptyset) - (\emptyset, \{1\}) - (\{1\}, \{1\}) - (\{1\}, [2]) - ([2], [2])$ is proper.

Remark 6.3. Note that in the definition of (very) proper decomposition we do not stipulate that $\mathcal{C}_i$ lies symmetrically inside Q_{n_i}, so the notion of a symmetric chain in $\prod_{i=1}^r \mathcal{C}_i$ might disagree with the notion of a symmetric chain in $\prod_{i=1}^r Q_{n_i}$. This is intentionally done to allow for certain constructions later on. We remark here that a necessary and sufficient condition for the notions to agree is that $\prod_{i=1}^r \mathcal{C}_i$ lies *symmetrically* inside $\prod_{i=1}^r Q_{n_i}$, meaning that the minimal and maximal elements are symmetric about the middle rank.

Proposition 6.4. *If we have a skipless chain $\mathcal{C} \subset \prod_{i=1}^{r} \mathcal{C}(\mathcal{S}_i)$ such that $\mathcal{C} \subset Q_n \cong Q_{n_1} \times \cdots \times Q_{n_r}$ is proper, then $\mathcal{C}$ intersects* every *symmetric chain $\mathcal{D} \subset \prod_{i=1}^{r} \mathcal{C}(\mathcal{T}_i)$ almost orthogonally.*

Proof. The cuboids $\prod_{i=1}^{r} \mathcal{C}(\mathcal{S}_i)$ and $\prod_{i=1}^{r} \mathcal{C}(\mathcal{T}_i)$ can only intersect in a rather specific way. Indeed, we have

$$\prod_{i=1}^{r} \mathcal{C}(\mathcal{S}_i) \cap \prod_{i=1}^{r} \mathcal{C}(\mathcal{T}_i) = \prod_{i=1}^{r} \left(\mathcal{C}(\mathcal{S}_i) \cap \mathcal{C}(\mathcal{T}_i)\right),$$

and by definition of $\mathcal{S}_i$ and $\mathcal{T}_i$ being almost orthogonal we have

$$\mathcal{C}(\mathcal{S}_i) \cap \mathcal{C}(\mathcal{T}_i) = \begin{cases} \{\emptyset, [n_i]\} & \text{if } \mathcal{C}(\mathcal{S}_i) \text{ and } \mathcal{C}(\mathcal{T}_i) \text{ are both} \\ & \text{maximal chains, and} \\ \text{at most one element in } Q_{n_i} & \text{otherwise.} \end{cases}$$

Hence, given distinct elements $x = (x_1, \ldots, x_r), y = (y_1, \ldots, y_r) \in \prod_{i=1}^{r} (\mathcal{C}(\mathcal{S}_i) \cap \mathcal{C}(\mathcal{T}_i))$, every coordinate i such that $x_i \neq y_i$ has the property that $\{x_i, y_i\} = \{\emptyset, [n_i]\}$. The properness of $\mathcal{C}$ then implies that at most one of x, y can lie in $\mathcal{C}$ unless $\{x, y\} = \{\emptyset, [n]\}$. Hence either $|\mathcal{C} \cap \mathcal{D}| \leq 1$ or $\mathcal{C}$ and $\mathcal{D}$ are both maximal chains and intersect precisely in their maximal and minimal elements, which by definition means that $\mathcal{C}$ and $\mathcal{D}$ are almost orthogonal. ∎

Our strategy for creating almost orthogonal decompositions is to give proper decompositions for as many product cuboids $\prod_{i=1}^{r} \mathcal{C}(\mathcal{S}_i) \subset Q_{n_1} \times \cdots \times Q_{n_r} \cong Q_n$ as possible to ensure that they cannot violate almost orthogonality with any other product decomposition, and then handle the remaining cuboids separately.

6.2. Universally proper and universally very proper chains. In this subsection, we create a framework for producing proper decompositions of cuboids $\prod_{i=1}^{r} \mathcal{C}(\mathcal{S}_i) \subset Q_{n_1} \times \cdots \times Q_{n_r} \cong Q_n$. Particularly, we want to

- create (very) proper decompositions of $\prod_{i=1}^{r} \mathcal{C}(\mathcal{S}_i)$ by working exclusively with the cuboid poset and not its embedding into the ambient $Q_{n_1} \times \cdots \times Q_{n_r}$, and
- have a "product construction" so that we can bootstrap (very) proper decompositions of lower dimensional cuboids to higher dimensional ones.

Suppose we have disjoint subsets $P_1, \ldots, P_\ell \subset \{1, \ldots, r\}$ and for each $1 \leq i \leq \ell$ a proper or very proper chain $\mathcal{C}_{P_i} \subset \prod_{j \in P_i} Q_{n_j}$. Our framework describes a sufficient condition for a skipless chain $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_{P_i} \subset \prod_{j \in \sqcup_{i=1}^{\ell} P_i} Q_{n_j}$ to be proper/very proper in terms of the poset embedding $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_{P_i}$ and the proper/very properness of the C_{P_i}, independent of the embeddings $C_{P_i} \subset \prod_{j \in P_i} Q_{n_j}$. Note that by Example 6.2 the chains $\mathcal{C}(\mathcal{S}_i) \subset Q_{n_i}$ themselves are proper/very proper.

With such a framework, our goal will be to reduce $\prod_{i=1}^{r} \mathcal{C}(\mathcal{S}_i)$ to proper chains as follows:

- Initialize a pool of cuboids to contain the cuboid $\prod_{i=1}^{r} \mathcal{C}(\mathcal{S}_i)$. Every cuboid in the pool will always be a product of the form $\prod_{i=1}^{m} \mathcal{C}_{R_i}$ lying symmetrically in Q_n with $\{R_1, \ldots, R_m\}$ a partition of $\{1, \ldots, n\}$ and $\mathcal{C}_{R_i} \subset \prod_{j \in R_i} Q_{n_j}$ a proper or very proper chain.
- From the pool of cuboids, choose a nontrivial subproduct $\prod_{i=1}^{\ell} \mathcal{C}_{P_i}$ of a cuboid in the pool where $P_1, \ldots, P_\ell$ are disjoint subsets of $\{1, \ldots, r\}$ and $C_{P_i} \subset \prod_{j \in P_i} Q_{n_j}$ is a proper or very proper chain. Decompose this subproduct into

symmetric proper or very proper chains in $\prod_{j\in\sqcup_{i=1}^{\ell} P_i} Q_{n_j}$, possibly leaving a product cuboid $\prod \mathcal{C}'_{P_i}$ of proper or very proper chains $\mathcal{C}'_{P_i} \subset \mathcal{C}_{P_i} \subset \prod_{j\in P_i} Q_{n_j}$ lying symmetrically as a remainder.
- Repeat until every cuboid in the pool is associated to the trivial partition, i.e., is a proper or very proper chain in $\prod_{i=1}^{r} Q_{n_i}$.

Great care must of course be taken to ensure that we don't produce indecomposable cuboids, and we note that depending on the proper/very properness of the $\mathcal{C}(\mathcal{S}_i)$'s and their lengths, such a decomposition may not even be possible.

For each cuboid formed as a product of proper and/or very proper chains, it will suffice to decompose one "universal" example of the same dimension into "universally proper/very proper" chains. Particularly in the above algorithm, we will only need to keep track of the dimensions of cuboids, the lengths of chains, and proper/very properness.

DEFINITION 6.5. *Given chain posets $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$, say that a map assigning each of these chain posets to the set {"universally proper," "universally very proper"} is a* valid designation *if chain posets of length ≤ 2 are assigned "universally very proper." We say that a designation of $\mathcal{C}$* agrees *with an embedding of $\mathcal{C}$ into a product of hypercubes if C is proper/very proper according to whether C is designated universally proper/very proper, respectively.*

DEFINITION 6.6. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ be chain posets with a valid designation. Then we say that a skipless chain $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_i$ is* universally proper *if for any pair of distinct elements $x = (x_1, \ldots, x_\ell), y = (y_1, \ldots, y_\ell) \in \mathcal{C}$, either*
- *all $\mathcal{C}_i$ are designated universally proper and x, y are the minimal and maximal elements in some order, or*
- *there exists a coordinate i such that $x_i \neq y_i$ and either $\mathcal{C}_i$ is universally very proper or $\{x_i, y_i\} \neq \{\min(C_i), \max(C_i)\}$.*

We say that a universally proper chain $\mathcal{C}$ is universally very proper *if it is not the case that all $\mathcal{C}_i$ are designated universally proper and $\mathcal{C}$ is a maximal chain. Say a symmetric chain decomposition of $\prod_{i=1}^{\ell} \mathcal{C}_i$ is* universally (very) proper *if each chain in the decomposition is* universally (very) proper.

Finally, we say that a designation of $\mathcal{C}$ universally agrees *with the embedding $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_i$ if C is universally proper/very proper according to whether C is designated universally proper/very proper, respectively.*

Remark 6.7. If we switch any nonempty subset of designations of the factors $\mathcal{C}_i$ from "universally proper" to "universally very proper" in a universally proper or universally very proper decomposition, we obtain a universally very proper decomposition with the new designations.

The following lemma relates the notions of universal (very) properness and (very) properness.

LEMMA 6.8. *Let $Q_n \cong \prod_{i=1}^{r} Q_{n_i}$, let $P_1, \ldots, P_\ell \subset \{1, \ldots, r\}$ be disjoint subsets, and let $\mathcal{C}_i \subset \prod_{j\in P_i} Q_{n_j}$ be skipless chains with valid designations agreeing with the embeddings such that every maximal chain $\mathcal{C}_i \subset \prod_{j\in P_i} Q_{n_j}$ is designated "universally proper." Then if a skipless chain $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_i$ is universally (very) proper, $\mathcal{C}$ is also a (very) proper chain in $\prod_{j\in\sqcup_{i=1}^{\ell} P_i} Q_{n_j}$.*

Proof. We first do the universally very proper case. By definition, the universally very proper chain $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_i$ has for any pair of distinct elements $x = (x_1, \ldots, x_\ell), y = (y_1, \ldots, y_\ell) \in \mathcal{C}$ a coordinate i such that $x_i \neq y_i$ and either $\mathcal{C}_i$ is designated universally very proper or $\{x_i, y_i\} \neq \{\min \mathcal{C}_i, \max \mathcal{C}_i\}$. In either case we have $x_i \neq y_i$ and $\{x_i, y_i\} \neq \{\emptyset, \prod_{j \in P_i}[n_j]\}$, and so by the properness of the embedding $\mathcal{C}_i \subset \prod_{j \in P_i} Q_{n_j}$ there exists a $j \in P_i$ such that $x_{i,j} \neq y_{i,j}$ and $\{x_{i,j}, y_{i,j}\} \neq \{\emptyset, [n_j]\}$. This verifies that $\mathcal{C}$ is proper. As every maximal $\mathcal{C}_i$ is designated universally proper, the definition of $\mathcal{C}$ being universally very proper precludes that it is a maximal chain in $\prod_{j \in \sqcup_{i=1}^{\ell} P_i} Q_{n_j}$, and hence $\mathcal{C}$ is universally very proper as desired.

Now, suppose that $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_i$ is universally proper and not universally very proper. Then we must have all $\mathcal{C}_i$ being designated universally proper and $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_i$ a maximal chain. For distinct $x, y \in \mathcal{C}$ with $\{x, y\} \neq \{\min \mathcal{C}, \max \mathcal{C}\}$, the identical argument to the universally very proper case shows there is a coordinate i and $j \in P_i$ such that $x_{i,j} \neq y_{i,j}$ and $\{x_i, y_i\} \neq \{\emptyset, [n_j]\}$. For $\{x, y\} = \{\min \mathcal{C}, \max \mathcal{C}\} = \{\prod_{i=1}^{\ell} \min \mathcal{C}_i, \prod_{i=1}^{\ell} \max \mathcal{C}_i\}$, note that as the designation is valid, $x_i \neq y_i$ for all i. Thus either there exists an i such that $x_i \neq y_i$ and $\{x_i, y_i\} \neq \{\emptyset, \prod_{j \in P_i}[n_j]\}$, in which case we may proceed as before, or $\{x_i, y_i\} = \{\emptyset, \prod_{j \in P_i}[n_j]\}$ for all i, which verifies that $\{x, y\}$ are the minimal/maximal elements of $\prod_{j \in \sqcup_{i=1}^{\ell} P_i} Q_{n_j}$. This shows that $\mathcal{C}$ is proper as desired.

Next, we show a compatibility result for universally proper and universally very proper chains.

LEMMA 6.9. *Suppose we have disjoint subsets $P_1 \ldots, P_\ell \subset \{1, \ldots, r\}$, and we have chain posets $\mathcal{C}$, $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ and for each $1 \leq i \leq \ell$ chain posets $\mathcal{C}_{i,j}$ with $j \in P_i$. Suppose further we have embeddings $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_i$ and $\mathcal{C}_i \subset \prod_{j \in P_i} \mathcal{C}_{i,j}$ as skipless chains, and valid designations of $\mathcal{C}$, the $\mathcal{C}_i$, and the $\mathcal{C}_{i,j}$ which universally agree with these embeddings. Then the designation of $\mathcal{C}$ universally agrees with the composite embedding $\mathcal{C} \subset \prod_{j \in \sqcup_{i=1}^{\ell} P_i} \mathcal{C}_{i,j}$.*

Proof. Suppose $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_i$ is universally very proper. Then for any pair of distinct elements $x = (x_1, \ldots, x_\ell), y = (y_1, \ldots, y_\ell) \in \mathcal{C}$, there is a coordinate i such that $x_i \neq y_i$ and either $\mathcal{C}_i$ is designated universally very proper or $\{x_i, y_i\} \neq \{\min \mathcal{C}_i, \max \mathcal{C}_i\}$. In either case, there exists a $j \in P_i$ such that $x_{i,j} \neq y_{i,j}$ and either $\mathcal{C}_{i,j}$ is designated universally very proper or $\{x_{i,j}, y_{i,j}\} \neq \{\min \mathcal{C}_{i,j}, \max \mathcal{C}_{i,j}\}$. Hence $\mathcal{C} \subset \prod_{j \in \sqcup_{i=1}^{\ell} P_i} \mathcal{C}_{i,j}$ is universally very proper, universally agreeing with any designation of $\mathcal{C}$.

Suppose now that $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_i$ is not universally very proper. Then it must have been designated universally proper, and so $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_i$ is universally proper. Consequently, as $\mathcal{C}$ is universally proper but not universally very proper, all $\mathcal{C}_i$ must be designated universally proper, so in particular all have length $\neq 1$, and $\mathcal{C} \subset \prod_{i=1}^{\ell} \mathcal{C}_i$ is a maximal chain. For $x, y \in \mathcal{C}$ with $\{x, y\} \neq \{\min \mathcal{C}, \max \mathcal{C}\}$ we can proceed verbatim to the universally very proper case, and hence it suffices to consider $\{x, y\} = \{\min \mathcal{C}, \max \mathcal{C}\}$. In $\prod_{j \in \sqcup_{i=1}^{\ell} P_i} \mathcal{C}_{i,j}$, either x, y are the maximal and minimal elements, in which case there is nothing to check, or else some $\mathcal{C}_i \subset \prod_{j \in P_i} \mathcal{C}_{i,j}$ is nonmaximal so must be universally very proper. As $x_i \neq y_i$ (since $\mathcal{C}_i$ has length $\neq 1$), we obtain from the universal very properness of $\mathcal{C}_i$ a $j \in P_i$ such that $x_{i,j} \neq y_{i,j}$ and either $\mathcal{C}_{i,j}$ is designated very proper or $\{x_{i,j}, y_{i,j}\} \neq \{\min \mathcal{C}_{i,j}, \max \mathcal{C}_{i,j}\}$. This means that $\mathcal{C} \subset \prod_{j \in \sqcup_{i=1}^{\ell} P_i} \mathcal{C}_{i,j}$ is universally proper, universally agreeing with its designation. ◻

The following fundamental lemma is very useful for producing universally proper and universally very proper decompositions, and we will refer to it as the "decomposition lemma."

Lemma 6.10 (decomposition lemma). *Suppose we have chain posets $\mathcal{C}_1$ and $\mathcal{C}_2$ with valid designations. Then the following statements are true:*

1. *If $\mathcal{C}_1, \mathcal{C}_2$ are both designated universally proper, then $\mathcal{C}_1 \times \mathcal{C}_2$ has a universally proper decomposition.*
2. *If $\mathcal{C}_1, \mathcal{C}_2$ are both designated universally very proper, then $\mathcal{C}_1 \times \mathcal{C}_2$ has a universally very proper decomposition.*
3. *If $\mathcal{C}_1$ is designated universally proper and $\mathcal{C}_2$ is a universally very proper chain of length ≥ 3, then $\mathcal{C}_1 \times \mathcal{C}_2$ has a universally very proper decomposition.*

In particular, suppose we have validly designated chain posets $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$.

- *If $\prod_{i=1}^{r-1} \mathcal{C}_i$ has a universally proper decomposition and $\mathcal{C}_r$ is a universally (very) proper chain of length ≥ 3, then $\prod_{i=1}^{r} \mathcal{C}_i$ has a universally (very) proper decomposition.*
- *If $\prod_{i=1}^{r-1} \mathcal{C}_i$ has a universally very proper decomposition and $\mathcal{C}_r$ is a universally very proper chain, then $\prod_{i=1}^{r} \mathcal{C}_i$ has a universally very proper decomposition.*

Remark 6.11. For the purposes of decomposition, it is always favorable to identify a chain as universally very proper rather than universally proper if possible. In particular, after a universally very proper decomposition all resulting chains should be designated as universally very proper, and after a universally proper decomposition all resulting chains, except the maximal chain if all factors were designated universally proper, should be designated universally very proper. This designation will clearly be valid unless the maximal chain in the latter case has length ≤ 2 (such an issue never arises for us, so this designation will always be valid).

In the context of the decomposition lemma, Lemma 6.10, the validity plays an important role, as if we are in a situation where one of our chains $\mathcal{C}_1$ and $\mathcal{C}_2$ is of length ≤ 2 and the other chain is designated universally proper, then we cannot proceed. This issue is precisely the issue identified in Remark 5.2, and we will often have to very carefully avoid it.

To prove the decomposition lemma, Lemma 6.10, we introduce what we call the "zigzag decomposition," a universally proper decomposition of a product of two universally proper chains of length ≥ 3.

Consider the decomposition of an $r \times s$ rectangle depicted in Figure 1 for $r = 4$, $s = 6$.

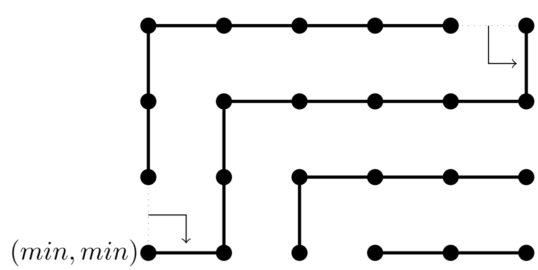

Fig. 1. *Zigzag decomposition for a 4×6 rectangle.*

Definition 6.12. *For $r, s \geq 3$, we say the zigzag decomposition of an $r \times s$ rectangle is the modification of the decomposition by 90-degree clockwise rotated L's as depicted in Figure 1. Specifically, we modify two edges of the two "leftmost" chains, at the smallest and largest elements.*

LEMMA 6.13. *Given a product of two chain posets validly designated universally proper, the zigzag decomposition is a universally proper decomposition.*

Proof. By the validity of the designation, both chains have length ≥ 3. If we draw the zigzag decomposition as in Figure 1, then the salient features which ensure it is a universally proper decomposition are that no chain horizontally connects a point on the left with the corresponding point on the right or vertically connects a point on the bottom with the corresponding point on the top, which can be checked by inspection. ∎

Proof of Lemma 6.10. For the second item, we can take any symmetric chain decomposition (as any two distinct elements of any chain have a distinct coordinate). The first and third items follow from Lemma 6.13. ∎

7. Products of two hypercubes and the notion of goodness. We now apply our strategy for creating almost orthogonal decompositions via a product construction to the product of two hypercubes. Most chain products have proper decompositions and can be safely ignored, and the remaining ones will have to be carefully dealt with—in fact, it turns out there are necessary and sufficient conditions on chains of minimal length to carry out our product construction, which turns out to be "goodness" as defined in Definition 2.3.

Suppose that we have k almost orthogonal symmetric chain decompositions $\{\mathcal{F}^1, \ldots, \mathcal{F}^k\}$ of Q_m, and $\{\mathcal{G}^1, \ldots, \mathcal{G}^k\}$ of Q_n, with $m, n \geq 2$ so no chain of length ≤ 2 is maximal.

Any product $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)$ with $\mathcal{C}(\mathcal{F}^j) \in \mathcal{F}^j$ and $\mathcal{C}(\mathcal{G}^j) \in \mathcal{G}^j$ has a universally proper decomposition provided the lengths of the two chains are ≥ 3 by the decomposition lemma, Lemma 6.10, which by Lemmas 6.8 and 6.9 is in fact a proper decomposition. Hence by Proposition 6.4, we may remove all such rectangles from further consideration. Furthermore, if neither of the chains is maximal, then both chains are very proper and the product has a very proper decomposition by Lemma 6.10, so these rectangles may be similarly removed.

We are thus left to consider rectangles that are the product of a maximal chain and a one- or two-element chain. It is impossible to do a proper symmetric chain decomposition of such rectangles, so we have to carefully consider their pairwise intersections.

Let

$$\epsilon_m = \begin{cases} 1 & \text{m is even and} \\ 2 & \text{otherwise,} \end{cases}$$

and similarly for ϵ_n so that ϵ_m and ϵ_n are the lengths of the smallest chains in Q_m and Q_n, respectively. Note that the only rectangles we have left are of sizes $\epsilon_m \times (n+1)$ and $(m+1) \times \epsilon_n$. Because $\{\mathcal{F}^j\}_{1 \leq j \leq k}$ and $\{\mathcal{G}^j\}_{1 \leq j \leq k}$ are each almost orthogonal families, an $\epsilon_m \times (n+1)$ rectangle arising as $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)$ and an $(m+1) \times \epsilon_n$ rectangle arising as $\mathcal{C}(\mathcal{F}^{j'}) \times \mathcal{C}(\mathcal{G}^{j'})$ with $j \neq j'$ will intersect in at most one element, hence any symmetric chain decompositions of these rectangles will have their chains intersect orthogonally, so we may ignore intersections between such rectangles.

We are left to consider separately intersections of $\epsilon_m \times (n+1)$ rectangles and intersections of $(m+1) \times \epsilon_n$ rectangles. We consider $\epsilon_m \times (n+1)$ rectangles; the case $(m+1) \times \epsilon_n$ is identical.

If $\epsilon_m = 1$, then any $\epsilon_m \times (n+1)$ rectangle $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)$ is already a symmetric chain, and the intersection of two such chains $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)$ and $\mathcal{C}(\mathcal{F}^{j'}) \times \mathcal{C}(\mathcal{G}^{j'})$ is precisely

$$(\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)) \cap (\mathcal{C}(\mathcal{F}^{j'}) \times \mathcal{C}(\mathcal{G}^{j'})) = (\mathcal{C}(\mathcal{F}^j) \cap \mathcal{C}(\mathcal{F}^{j'})) \times \{\emptyset, [m] \times [n]\}.$$

Thus the chains will always intersect almost orthogonally if and only if the one-element chains from the $\mathcal{F}^j$ for varying j are all distinct.

If $\epsilon_m = 2$, then there are two ways of decomposing a $2 \times (n+1)$ rectangle into symmetric chains.

Definition 7.1. *Given a $2 \times r$ rectangle poset, we define the* bottom decomposition *and* top decomposition *to be the two possible decompositions according to Figure* 2.

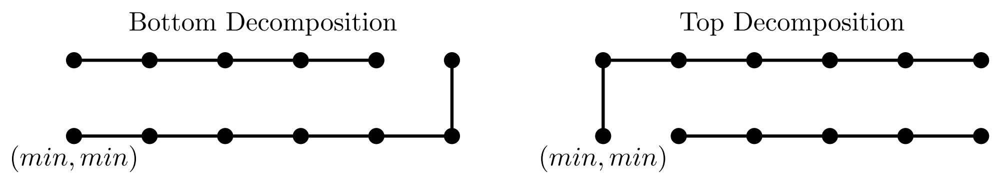

Fig. 2. *Bottom and top decompositions of a $2 \times r$ rectangle poset, respectively.*

Consider two $2 \times (n+1)$ rectangles $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)$ and $\mathcal{C}(\mathcal{F}^{j'}) \times \mathcal{C}(\mathcal{G}^{j'})$ with $j \neq j'$. Each of these rectangles can be decomposed using the bottom or the top decomposition, but we want to be careful to ensure that the two resulting chains in $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)$ intersect the two resulting chains in $\mathcal{C}(\mathcal{F}^{j'}) \times \mathcal{C}(\mathcal{G}^{j'})$ almost orthogonally.

We have three cases to consider:

1. If the two-element chains $\mathcal{C}(\mathcal{F}^j)$ and $\mathcal{C}(\mathcal{F}^{j'})$ do not intersect, then the rectangles $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)$ and $\mathcal{C}(\mathcal{F}^{j'}) \times \mathcal{C}(\mathcal{G}^{j'})$ are disjoint so regardless of whether top or bottom decompositions are chosen, the resulting chains will in fact be disjoint.
2. If the two-element chains $\mathcal{C}(\mathcal{F}^j)$ and $\mathcal{C}(\mathcal{F}^{j'})$ intersect in their common maximal element, then the chains from the decompositions of $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)$ and $\mathcal{C}(\mathcal{F}^{j'}) \times \mathcal{C}(\mathcal{G}^{j'})$ intersect almost orthogonally if and only if they are not both top decompositions.
3. If the two-element chains $\mathcal{C}(\mathcal{F}^j)$ and $\mathcal{C}(\mathcal{F}^{j'})$ intersect in their common minimal element, then the chains from the decompositions of $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)$ and $\mathcal{C}(\mathcal{F}^{j'}) \times \mathcal{C}(\mathcal{G}^{j'})$ intersect almost orthogonally if and only if they are not both bottom decompositions.

Note that each two-element chain $\mathcal{C}(\mathcal{F}^j)$ is part of a unique $2 \times (n+1)$ rectangle $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)$ since the maximal chain $\mathcal{C}(\mathcal{G}^j)$ is unique in $\mathcal{T}^j$. With this in mind, construct an oriented directed graph whose vertex set is the union of all two-element chains $\mathcal{C}(\mathcal{F}^j)$ for all j, with edges corresponding to the two-element chains $\mathcal{C}(\mathcal{F}^j)$ themselves, oriented from minimal element to maximal element if the unique $2 \times (n+1)$ rectangle corresponding to $\mathcal{C}(\mathcal{F}^j)$ is part of has a bottom decomposition, and otherwise from the maximal element to the minimal element if the two-element chain is part of a top decomposition.

Definition 7.2. *Call a graph good if it can be given an orientation such that the out-degree of every vertex is at most* 1.

Lemma 7.3. *A graph is good if and only if every component has at most one cycle (equivalently, every component is either a tree, or a tree union an edge).*

Proof. If a component has no cycle, then choose a root vertex and direct all edges toward it. If a component has a unique cycle, give the cycle an orientation, then direct all remaining arrows toward the cycle. If a component has two cycles, then it is easy to see the cycles would have to be disjoint, and then one can check there is no way to orient a path between the two cycles. ∎

The above discussion shows that the definition of goodness from Definition 2.3 is precisely what we need to make the following proposition true.

PROPOSITION 7.4. *Let $m, n \geq 2$, and suppose Q_m and Q_n each have a family of k almost orthogonal symmetric chain decompositions $\{\mathcal{F}^1, \ldots, \mathcal{F}^k\}$ and $\{\mathcal{G}^1, \ldots, \mathcal{G}^k\}$, respectively. Then $Q_m \times Q_n = Q_{m+n}$ has an almost orthogonal symmetric chain decomposition arising from decomposing the rectangles $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j)$ with $\mathcal{C}(\mathcal{F}^j) \in \mathcal{F}^j$ and $\mathcal{C}(\mathcal{G}^j) \in \mathcal{G}^j$ into symmetric chains if and only if the two families of decompositions are both good.*

We have checked that the decompositions of Q_5 and Q_7 provided in section 4 are good when we introduced them. Note that the product of two hypercubes with good families of almost orthogonal symmetric chain decompositions may not have a good family of almost orthogonal symmetric chain decompositions, so we can't use Proposition 7.4 to produce decompositions of arbitrarily large hypercubes yet. However, we still have the following.

COROLLARY 7.5. *Q_{10}, Q_{12}, Q_{14} all have three almost orthogonal symmetric chain decompositions.*

8. Universally proper decompositions. In this section we prove the following theorem.

THEOREM 8.1. *Suppose we have a product of chain posets $\prod_{i=1}^{\ell} \mathcal{C}_i$ with the valid designation*

$$C_i \mapsto \begin{cases} \textit{universally very proper} & 1 \leq |C_i| \leq 4 \textit{ and} \\ \textit{universally proper} & 5 \leq |C_i|. \end{cases}$$

Then there is a universally proper decomposition of $\prod_{i=1}^{\ell} \mathcal{C}_i$ except possibly in one of the following two cases when $r \geq 2$:

- *Exactly one $\mathcal{C}_i$ is of length ≥ 5, and all others have length ≤ 2.*
- *All $\mathcal{C}_i$ have length either 1 or ≥ 5, not all have length 1, and not all have length ≥ 5.*

Remark 8.2. The valid designation from Theorem 8.1 is particularly useful for proving Theorems 3.3, 3.5, and 3.6, because maximal chains in a product of hypercubes of dimension ≥ 4 (even a trivial product with one hypercube) have length ≥ 5.

Example 8.3. Let's check that in the second excluded case of Theorem 8.1 we can't produce a universally proper decomposition. Suppose that we could. As not all $\mathcal{C}_i$ are designated universally proper, every chain in the resulting universally proper decomposition is automatically universally very proper. But then consider the maximal chain and its minimal element x and maximal element y. As these are distinct elements of a universally very proper chain, there exists a coordinate i such that $x_i \neq y_i$ and either $\mathcal{C}_i$ is designated universally very proper or $\{x_i, y_i\} \neq \{\min \mathcal{C}_i, \max \mathcal{C}_i\}$. But $\mathcal{C}_i$ designated universally very proper implies it has length 1, contradicting that $x_i \neq y_i$, and for all other factors we have $\{x_i, y_i\} = \{\min \mathcal{C}_i, \max \mathcal{C}_i\}$ by the definition of x and y. Hence, we cannot have a universally proper decomposition.

Note that a chain of length 1 is universally very proper, so by the decomposition lemma, Lemma 6.10, the product with any universally very proper chain decomposition has a universally very proper chain decomposition. Hence, it suffices to show the following.

LEMMA 8.4. *With the hypotheses of Theorem* 8.1*, if furthermore all* $\mathcal{C}_i$ *have length* ≥ 2*, then we have a universally proper decomposition except possibly when* $r \geq 2$*, exactly one* $\mathcal{C}_i$ *is of length* ≥ 5*, and all others have length* 2*.*

The remainder of the section is devoted to the proof of this proposition, so in particular we are assuming $\mathcal{C}_i$ has length ≥ 2 for the remainder of this section.

8.1. Lemma 8.4: All $\mathcal{C}_i$ of length ≥ 3. First, we show that it suffices to handle the case when all $\mathcal{C}_i$ have length ≥ 5. Indeed, suppose we know how to do this case. If all $\mathcal{C}_i$ are of length ≥ 5, then we are done. Otherwise, suppose without loss of generality that $\mathcal{C}_1$ is a universally very proper chain, of length either 3 or 4. Then take a universally proper decomposition of all chains of length ≥ 5, and by the decomposition lemma, Lemma 6.10, we can decompose the further product with $\mathcal{C}_1$ into a universally very proper decomposition; then as all remaining $\mathcal{C}_i$ are universally very proper we may repeatedly apply Lemma 6.10 to produce a universally very proper decomposition of the product of the resulting chains with the remaining $\mathcal{C}_i$.

Hence, suppose that all $\mathcal{C}_i$ have length ≥ 5. We proceed by induction on r. For $r = 1$ there is nothing to prove and for $r = 2$ we may use Lemma 6.10, so suppose now $r > 2$. Take $\mathcal{C}_{r-1} \times \mathcal{C}_r$, and denote by s the length of $\mathcal{C}_{r-1}$ and by t the length of $\mathcal{C}_r$. Then decompose $\mathcal{C}_{r-1} \times \mathcal{C}_r$ into the two modified chains $\mathcal{C}', \mathcal{C}''$ in the zigzag decomposition of lengths $s+t-1, s+t-3$, respectively, and the remaining $(s-2)\times(t-2)$ rectangle R lying symmetrically inside $C_{r-1} \times C_r$. Then $\mathcal{C}' \subset \mathcal{C}_{r-1} \times \mathcal{C}_r$ is universally proper, $\mathcal{C}'' \subset \mathcal{C}_{r-1} \times \mathcal{C}_r$ is universally very proper, and

$$R = (\mathcal{C}_{r-1} \setminus \text{top 2 elements})) \times (\mathcal{C}_r \setminus \text{bottom 2 elements}),$$

where $(\mathcal{C}_{r-1} \setminus \text{top 2 elements}) \subset \mathcal{C}_{r-1}$ and $(\mathcal{C}_r \setminus \text{bottom 2 elements}) \subset \mathcal{C}_r$ are universally very proper chains of length ≥ 3.

By induction, as $\mathcal{C}'$ and $\mathcal{C}''$ have length ≥ 5 we have universally proper decompositions of $(\prod_{i=1}^{r-2} \mathcal{C}_i) \times \mathcal{C}'$ and $(\prod_{i=1}^{r-2} \mathcal{C}_i) \times \mathcal{C}''$, and $(\prod_{i=1}^{r-2} \mathcal{C}_i) \times (\mathcal{C}_{r-1} \setminus \text{top 2 elements}) \times (\mathcal{C}_r \setminus \text{bottom 2 elements})$ has a universally very proper decomposition as follows. First, apply induction to the first $r-2$ factors to yield a universally proper decomposition, then apply Lemma 6.10 twice, first with the resulting decomposition and $(C_{r-1} \setminus \text{top 2 elements})$, and then with the resulting universally very proper decomposition and $(C_r \setminus \text{bottom 2 elements})$.

8.2. Lemma 8.4: $2 \times s \times t$ cuboids. Consider chain posets $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ with lengths $2, s, t$, respectively, such that either $s = t = 2$ or $s, t \geq 3$.

If $s = t = 2$, then this is the product of three universally very proper chains, and the decomposition lemma, Lemma 6.10, shows that this has a universally very proper decomposition.

If $s, t \geq 3$ but one of $s, t \leq 4$, suppose without loss of generality that $s \leq 4$. Then $\mathcal{C}_2$ is universally very proper of length ≥ 3 so $\mathcal{C}_2 \times \mathcal{C}_3$ has a universally proper decomposition by Lemma 6.10. Then since $\mathcal{C}_1$ is universally very proper, the product thus has a universally very proper decomposition by Lemma 6.10.

If $s, t \geq 5$, then take the decomposition of $\mathcal{C}_1 \times \mathcal{C}_2$, as depicted in Figure 3.

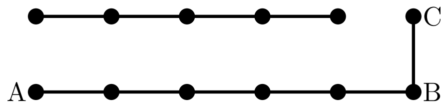

FIG. 3. *Decomposition of $\mathcal{C}_1 \times \mathcal{C}_2$ rectangle with longer chain labeled.*

The smaller chain is very proper of length ≥ 3, so taking the product with $\mathcal{C}_3$ is handled by the decomposition lemma, Lemma 6.2. For the longer chain, we decompose the product with $\mathcal{C}_3$ as in Figure 4. Only three chains were modified from a decomposition into clockwise rotated L's, namely, the three "leftmost" chains. Graphically, the salient features that need to be checked in Figure 4 to ensure that we have a very proper decomposition are that the bottom left A is not connected to the upper right B, that no chain connects a point on the left side horizontally to the corresponding point on the right side, and that no point on the bottom side connects vertically to the corresponding point on the line segment connecting the two B's.

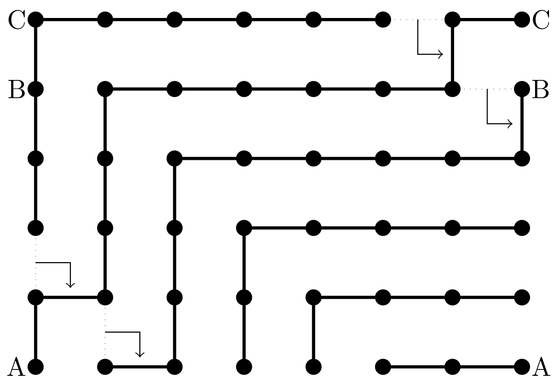

FIG. 4. *Decomposition of (longer chain from Figure 2)$\times\mathcal{C}_3$.*

8.3. Lemma 8.4: $2 \times s \times t \times u$ cuboids. Consider chain posets $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ with lengths $2 \leq s \leq t \leq u$ respectively such that we don't have both $s = t = 2$ and $u \geq 5$. If $s \leq 4$, then $\mathcal{C}_2$ is universally very proper, and by Lemma 6.10, we may take the product of $\mathcal{C}_2$ with the universally very proper decomposition from subsection 8.2 of $\mathcal{C}_1 \times \mathcal{C}_3 \times \mathcal{C}_4$ to get a universally very proper decomposition.

Finally, suppose that $s, t, u \geq 5$. Take $\mathcal{C}_3 \times \mathcal{C}_4$ and partially decompose it as the two modified universally proper chains $\mathcal{C}', \mathcal{C}''$ from the zigzag decomposition of lengths $t + u - 1, t + u - 3 \geq 5$, respectively, and the rectangle

$$R = (\mathcal{C}_3 \setminus \text{top 2 elements}) \times (\mathcal{C}_4 \setminus \text{bottom 2 elements}),$$

the product of universally very proper chains $(\mathcal{C}_3 \setminus \text{top 2 elements}) \subset \mathcal{C}_3$ and $(\mathcal{C}_4 \setminus$ bottom 2 elements$) \subset \mathcal{C}_4$, which lies symmetrically in $C_3 \times C_4$. Then by the previous subsection, we may produce universally very proper decompositions of $\mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}'$ and $\mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}''$, and for $\mathcal{C}_1 \times \mathcal{C}_2 \times (\mathcal{C}_3 \setminus \text{top 2 elements}) \times (\mathcal{C}_4 \setminus \text{bottom 2 elements})$, we apply Lemma 6.10 to get a universally very proper decomposition of $\mathcal{C}_2 \times (\mathcal{C}_3 \setminus$ top 2 elements), and since $\mathcal{C}_1$ and ($\mathcal{C}_4 \setminus$ bottom 2 elements) $\subset \mathcal{C}_4$ are universally very proper we may apply Lemma 6.10 two more times to obtain a universally very proper decomposition of the product.

8.4. Lemma 8.4: Remaining cases. By subsection 8.1, in all remaining cases there is at least one chain of length 2 which we may suppose is $\mathcal{C}_1$.

If all $\mathcal{C}_i$ have length ≤ 4, then they are all universally very proper and we may repeatedly apply the decomposition lemma, Lemma 6.10, to obtain a universally very proper decomposition.

If there is exactly one $\mathcal{C}_i$ of length ≥ 5, without loss of generality $\mathcal{C}_2$, then as all remaining chains can't have length 2 by the hypothesis of Lemma 8.4 there must be a chain of length 3 or 4, without loss of generality $\mathcal{C}_3$, and then we may obtain a universally very proper decomposition of $\mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$ by subsection 8.2, and then as the remaining $\mathcal{C}_i$ are universally very proper we obtain a universally very proper decomposition by repeated applications of Lemma 6.10.

If there are exactly two $\mathcal{C}_i$ of length ≥ 5, without loss of generality $\mathcal{C}_2$ and $\mathcal{C}_3$, then by subsection 8.2 we obtain first a universally very proper decomposition of $\mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$, and then we obtain a universally very proper decomposition of the entire product $\prod_{i=1}^{r} \mathcal{C}_i$ by repeatedly applying Lemma 6.10.

Suppose now there are at least three $\mathcal{C}_i$ of length ≥ 5. By subsection 8.1, we obtain a universally proper decomposition of all but two of these $\mathcal{C}_i$ of length ≥ 5. We now have to decompose the following cuboids:

- The product of two universally proper chains of length ≥ 5 with $\mathcal{C}_1$ and some universally very proper chains. Here we obtain a universally very proper decomposition of the first three factors by subsection 8.2, and then we repeatedly apply Lemma 6.10 with the remaining universally very proper chains.
- The product of three universally proper chains of length ≥ 5 with $\mathcal{C}_1$ and some universally very proper chains. Here we obtain a universally very proper decomposition of the first four factors by subsection 8.3, and then we repeatedly apply Lemma 6.10 with the remaining universally very proper chains.

Lemma 8.4, and hence Theorem 8.1, now follows.

9. Products of a small number of hypercubes. In this section we prove the remaining auxilliary results needed in the proof of our main results. By Lemmas 6.8 and 6.9, universally proper decompositions produced from applications of Theorem 8.1 are in fact proper decompositions. Theorem 8.1 will always be applicable because we will be working in products of hypercubes Q_{n_i} with $n_i \geq 4$ (see Remark 8.2). We also frequently omit explicit reference to Proposition 6.4, which allows us to ignore proper chains for the purposes of checking almost orthogonality.

PROPOSITION 9.1. *Let $r \geq 2$, and suppose for each $1 \leq i \leq r$ we have an integer $n_i \geq 2$ and a family of k almost orthogonal symmetric chain decompositions $\{\mathcal{F}_i^j\}_{1 \leq j \leq k}$ of Q_{n_i}, and suppose that the family of decompositions for Q_{n_i} is good for some $1 \leq i \leq r-1$. Then there is a symmetric chain decomposition of each cuboid $\prod_{i=1}^{r} \mathcal{C}(\mathcal{F}_i^k)$ of dimension $2 \times \cdots \times 2 \times (n_r+1)$ with $\mathcal{C}(\mathcal{F}_i^k) \in \mathcal{F}_i^k$ such that any symmetric chain from a decomposition of some $\prod_{i=1}^{r} \mathcal{C}(\mathcal{F}_i^k)$ and any symmetric chain from a decomposition of some $\prod_{i=1}^{r} \mathcal{C}(\mathcal{F}_i^{k'})$ with $k' \neq k$ are orthogonal.*

Proof. Without loss of generality we may assume that the goodness applies to Q_{n_i} for $i = r-1$. We decompose the cuboids $\prod_{i=1}^{r} \mathcal{C}(\mathcal{F}_i^k)$ as follows. By the goodness of $Q_{n_{r-1}}$, for each choice of two-element chain $\mathcal{C}(\mathcal{F}^k_{r-1})$ and $\mathcal{C}(\mathcal{F}^k_r)$ the unique maximal chain in $\mathcal{F}^k_r$, we may choose symmetric chain decompositions of each $\mathcal{C}(\mathcal{F}^k_{r-1})\times\mathcal{C}(\mathcal{F}^k_r)$ so that the chains from the decomposition of $\mathcal{C}(\mathcal{F}^k_{r-1})\times\mathcal{C}(\mathcal{F}^k_r)$ and from $\mathcal{C}(\mathcal{F}^{k'}_{r-1})\times\mathcal{C}(\mathcal{F}^{k'}_r)$ intersect orthogonally whenever $k\neq k'$. Hence the products of the chains from $\mathcal{C}(\mathcal{F}^k_{r-1})\times\mathcal{C}(\mathcal{F}^k_r)$ with any product of two-element chains $\prod_{i=1}^{r-2}\mathcal{C}(\mathcal{F}^k_i)\in\prod_{i=1}^{r-2}\mathcal{F}^k_i$ and the products of the chains from $\mathcal{C}(\mathcal{F}^{k'}_{r-1})\times\mathcal{C}(\mathcal{F}^{k'}_r)$ with any product of two-element chains $\prod_{i=1}^{r-2}\mathcal{C}(\mathcal{F}^{k'}_i)\in\prod_{i=1}^{r-2}\mathcal{F}^{k'}_i$ will intersect in at most one element as by the almost orthogonality of $\{\mathcal{F}^k_i,\mathcal{F}^{k'}_i\}$ we have $|\mathcal{C}(\mathcal{F}^k_i)\cap\mathcal{C}(\mathcal{F}^{k'}_i)|=1$ for $1\le i\le r-2$. Hence any symmetric chain decompositions of these resulting products yield orthogonal symmetric chains as desired. ∎

9.1. Product of three hypercubes, one odd and two good, or three good. The goal of this subsection is to prove the following.

PROPOSITION 9.2. *Let $m,n,p\ge 4$, and suppose Q_m, Q_n, Q_p have $k\ge 3$ almost orthogonal symmetric chain decompositions $\{\mathcal{F}^j\}_{1\le j\le k}$, $\{\mathcal{G}^j\}_{1\le j\le k}$, and $\{\mathcal{H}^j\}_{1\le j\le k}$, respectively, two of these families are good, and the remaining family is either good or is associated to an odd dimensional hypercube. Then Q_{m+n+p} has k almost orthogonal symmetric chain decompositions.*

Proof. By Theorem 8.1, all cuboids $\mathcal{C}(\mathcal{F}^j)\times\mathcal{C}(\mathcal{G}^j)\times\mathcal{C}(\mathcal{H}^j)$ have proper decompositions except possibly if the cuboids have the following dimensions. In the following, let $\epsilon_1,\epsilon_2,\epsilon_3$ be the lengths of the smallest chains in Q_m,Q_n,Q_p, respectively:

- cuboids $(m+1)\times\epsilon_2\times\epsilon_3$, $\epsilon_1\times(n+1)\times\epsilon_3$, and $\epsilon_1\times\epsilon_2\times(p+1)$,
- cuboids $1\times(n+1)\times(p+1)$, $(m+1)\times 1\times(p+1)$, and $(m+1)\times(n+1)\times 1$ if they exist (i.e., if $\epsilon_1=1$, $\epsilon_2=1$, or $\epsilon_3=1$, respectively).

We note now that the conditions in Proposition 9.2 imply that any factor corresponding to an even dimensional hypercube is good.

Suppose we have a product $\mathcal{C}(\mathcal{F}^j)\times\mathcal{C}(\mathcal{G}^j)\times\mathcal{C}(\mathcal{H}^j)$ which is of dimension $(m+1)\times(n+1)\times 1$. By the decomposition lemma, Lemma 6.10, we can create a universally proper decomposition of $\mathcal{C}(\mathcal{F}^j)\times\mathcal{C}(\mathcal{G}^j)$, and after taking each chain $\times\mathcal{C}(\mathcal{H}^j)$ then only the maximal chain is not very proper, the properness condition failing only for the pair of minimal and maximal elements, so the maximal chain could potentially intersect another symmetric chain from another cuboid in the two elements $(\emptyset,\emptyset,\mathcal{C}(\mathcal{H}^j))$ and $([m],[n],\mathcal{C}(\mathcal{H}^j))$. If for some $j'\neq j$ we have a cuboid $\mathcal{C}(\mathcal{F}^{j'})\times\mathcal{C}(\mathcal{G}^{j'})\times\mathcal{C}(\mathcal{H}^{j'})$ of one of the dimensions that we're considering, the only cuboid dimension which has a chance of intersecting in these two elements is another $(m+1)\times(n+1)\times 1$ cuboid. But because we have a one-element chain $\mathcal{C}(\mathcal{H}^j)$ in the last factor, Q_p must be a good even dimensional hypercube, so in particular $\mathcal{C}(\mathcal{H}^j)\neq\mathcal{C}(\mathcal{H}^{j'})$ forcing the cuboids to in fact be disjoint.

We reason similarly if $\mathcal{C}(\mathcal{F}^j)\times\mathcal{C}(\mathcal{G}^j)\times\mathcal{C}(\mathcal{H}^j)$ is of dimension $(m+1)\times 1\times(p+1)$ or $1\times(n+1)\times(p+1)$, so we can restrict our attention exclusively to cuboids $\mathcal{C}(\mathcal{F}^j)\times\mathcal{C}(\mathcal{G}^j)\times\mathcal{C}(\mathcal{H}^j)$ with dimensions $(m+1)\times\epsilon_2\times\epsilon_3$, $\epsilon_1\times(n+1)\times\epsilon_3$, and $\epsilon_1\times\epsilon_2\times(p+1)$.

Note that if the maximal chain is in a different factor for another such cuboid $\mathcal{C}(\mathcal{F}^{j'})\times\mathcal{C}(\mathcal{G}^{j'})\times\mathcal{C}(\mathcal{H}^{j'})$ with $j'\neq j$, then by the almost orthogonality of the decompositions the cuboids intersect in at most one element, so any symmetric chain decompositions of the cuboids will have their respective chains intersect orthogonally. Hence we only have to deal with the intersection properties of those cuboids with the maximal chain in the same factor. This then fixes the dimensions of the cuboids under consideration to be the same.

If one of the factors is length 1, then that factor corresponds to an even dimensional, hence good, hypercube, so arguing like before we conclude that all such cuboids are disjoint. Otherwise, such a cuboid has at least one of its two length-2 factors in a good hypercube, say, Q_n (we only use goodness in this factor now). By symmetry, the only case we have to deal with is cuboids of dimension $2 \times 2 \times (p+1)$. Proposition 9.1 implies that we can decompose such cuboids so that the resulting chains intersect orthogonally, as desired. ◻

9.2. Product of four odd hypercubes, two good. The goal of this subsection is to prove the following.

PROPOSITION 9.3. *Take $m, n, p, q \geq 4$ odd. Suppose Q_m, Q_n, Q_p, and Q_q have families of $k \geq 3$ almost orthogonal symmetric chain decompositions $\{\mathcal{F}^j\}_{1\leq j\leq k}$, $\{\mathcal{G}^j\}_{1\leq j\leq k}$, $\{\mathcal{H}^j\}_{1\leq j\leq k}$, and $\{\mathcal{K}^j\}_{1\leq j\leq k}$, respectively, and two of these families are good. Then $Q_{m+n+p+q}$ has k almost orthogonal symmetric chain decompositions.*

Proof. By Theorem 8.1, all cuboids $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j) \times \mathcal{C}(\mathcal{H}^j) \times \mathcal{C}(\mathcal{K}^j)$ have proper decompositions except possibly those with three factors of length 2, and one factor maximal. By almost orthogonality, two such cuboids can intersect in more than one element only if the maximal chain is in the same factor. We will show how to handle $2 \times 2 \times 2 \times (q+1)$ cuboids; the other cases are similarly dealt with. As two of the decompositions are good, without loss of generality assume Q_p (the third factor) is good. Proposition 9.1 implies that we can decompose such cuboids so that the resulting chains intersect orthogonally, as desired. ◻

9.3. Arbitrarily many good even hypercubes. The goal of this subsection is to prove the following.

PROPOSITION 9.4. *For $1 \leq i \leq r$, suppose we have $k \geq 3$ good almost orthogonal symmetric chain decompositions $\{\mathcal{F}_i^j\}_{1\leq j\leq k}$ of Q_{n_i} with all $n_i \geq 4$ even. Then $Q_{n_1+\cdots+n_r}$ has k almost orthogonal symmetric chain decompositions.*

Proof. By Theorem 8.1, all cuboids $\prod_{i=1}^r \mathcal{C}(\mathcal{F}_i^j)$ have proper decompositions except possibly those which are the products of chains of maximal size and minimal size (size 1) and contain at least one chain of minimal size.

For each of these, by Theorem 8.1, we may create a universally proper decomposition of the factors containing the chains of maximal size. After taking the product with the one-element chains, every chain remains universally very proper except the maximal chain in the cuboid, and the only two elements the maximal chain could have in common with any other chain are its maximal and minimal elements as this is the only pair of elements that universal properness fails at. Hence, if two chains intersected in two elements, it would be two maximal chains intersecting in their common top and bottom elements, forcing the cuboids to have the same dimensions. But then they share a factor with a one-element chain, which by the goodness of the factor containing the one-element chain forces the cuboids to be disjoint. ◻

9.4. Four hypercubes, one odd and three good even. The goal of this subsection is to prove the following, the last case needed before we prove our main results.

PROPOSITION 9.5. *Take $m \geq 4$ odd and $n, p, q \geq 2$ even. Suppose Q_m, Q_n, Q_p, and Q_q have families of $k \geq 3$ almost orthogonal symmetric chain decompositions $\{\mathcal{F}^j\}_{1\leq j\leq k}, \{\mathcal{G}^j\}_{1\leq j\leq k}$, $\{\mathcal{H}^j\}_{1\leq j\leq k}$, and $\{\mathcal{K}^j\}_{1\leq j\leq k}$, with the decompositions of Q_n, Q_p, and Q_q good. Then $Q_{m+n+p+q}$ has k almost orthogonal symmetric chain decompositions.*

Proof. By Theorem 8.1, all cuboids $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j) \times \mathcal{C}(\mathcal{H}^j) \times \mathcal{C}(\mathcal{K}^j)$ have proper decompositions except possibly those formed by products of maximal and minimal length chains.

Consider first when the factor in Q_m is a two-element chain. By Theorem 8.1, every product $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j) \times \mathcal{C}(\mathcal{H}^j) \times \mathcal{C}(\mathcal{K}^j)$ has a proper decomposition except possibly those which have dimensions $2 \times (n+1) \times 1 \times 1$, $2 \times 1 \times (p+1) \times 1$, or $2 \times 1 \times 1 \times (q+1)$. Any two such cuboids $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j) \times \mathcal{C}(\mathcal{H}^j) \times \mathcal{C}(\mathcal{K}^j)$ and $\mathcal{C}(\mathcal{F}^{j'}) \times \mathcal{C}(\mathcal{G}^{j'}) \times \mathcal{C}(\mathcal{H}^{j'}) \times \mathcal{C}(\mathcal{K}^{j'})$ with $j \neq j'$ have a one-element chain in the same factor, which by the goodness of the corresponding hypercube forces them to be disjoint. We will shortly return to consider their intersections with cuboids from the next case.

Assume now that the factor in Q_m is a maximal chain, so every factor in $\mathcal{C}(\mathcal{F}^j) \times \mathcal{C}(\mathcal{G}^j) \times \mathcal{C}(\mathcal{H}^j) \times \mathcal{C}(\mathcal{K}^j)$ is maximal or length 1. Theorem 8.1 gives us a proper decomposition when all of the factors are maximal or minimal, so the remaining cuboids have dimensions $(m+1)\times(n+1)\times 1\times 1, (m+1)\times 1\times(p+1)\times 1, (m+1)\times 1\times 1\times(q+1), (m+1)\times(n+1)\times(p+1)\times 1, (m+1)\times(n+1)\times 1\times(q+1), (m+1)\times 1\times(p+1)\times(q+1)$. Identical to the proof of Proposition 9.4, any two such cuboids have no problems with their pairwise intersections when we properly decompose the non-1 factors.

Thus we only have to consider the intersections of the cuboids in the first case without universally proper decompositions with the cuboids in the second case without universally proper decompositions. By almost orthogonality, the maximal length chain in the cuboid from the first case must be in the same factor as a maximal length chain in the cuboid from the second case for the cuboids to have intersection of size at least 2. But then the cuboids must share a factor with a one-element chain, forcing disjointness by the goodness of the corresponding hypercube. ∎

10. Proof of Theorem 3.3, Corollary 3.4, and Theorem 3.5.

In this section, we will prove Theorem 3.5, from which we will immediately deduce Theorem 3.3 and Corollary 3.4. The considerations from the beginning of section 9 still apply.

Proof of Theorem 3.5. We have the following transformations of products of hypercubes, each of which has $k \geq 3$ almost orthogonal symmetric chain decompositions (where odd and even refer to the dimension of the corresponding hypercube and good refers to the collection of decompositions):

$$\begin{aligned}
\text{1 good odd, 1 good even} &\to \text{1 odd (Proposition 7.4)},\\
\text{2 good odd, 1 odd} &\to \text{1 odd (Proposition 9.2)},\\
\text{2 good even, 1 odd} &\to \text{1 odd (Proposition 9.2)},\\
\text{3 good even, 1 odd} &\to \text{1 odd (Proposition 9.5)},\\
\text{2 good odd} &\to \text{1 even (Proposition 7.4)},\\
\text{2 good odd, 1 good even} &\to \text{1 even (Proposition 9.2)},\\
\text{1 good even, 1 good odd, 1 odd} &\to \text{1 even (Proposition 9.2)},\\
\text{2 good odd, 2 odd} &\to \text{1 even (Proposition 9.3)},\\
k \geq 1 \text{ good even} &\to \text{1 even (Proposition 9.4)}.
\end{aligned}$$

Our plan is to repeatedly use these transformations to replace subsets of hypercube factors in $Q_{n_1} \times \cdots \times Q_{n_k}$ with the corresponding product hypercube until we reduce down to $Q_{n_1+\ldots+n_k}$. Note that to transform an even dimensional hypercube it needs to be good, so if we transform some hypercubes into one even dimensional hypercube, this must be the last transformation applied since we have no guarantee of its goodness.

- Suppose there are no odd dimensional hypercubes. Then every hypercube is a good even dimensional hypercube, and we may apply

$$k \geq 1 \text{ good even} \rightarrow 1 \text{ even (Proposition 9.4).}$$

- Suppose there is exactly one odd dimensional hypercube. If there is an odd number of even dimensional hypercubes (all of which are good), then apply once the transformation

$$1 \text{ good odd, } 1 \text{ good even} \rightarrow 1 \text{ odd (Proposition 7.4).}$$

  Regardless of how many even dimensional hypercubes there were to start, we now have

$$\text{even number of good even, } 1 \text{ odd.}$$

  But now we can repeatedly apply the transformation

$$2 \text{ good even, } 1 \text{ odd} \rightarrow 1 \text{ odd (Proposition 9.2)}$$

  to reduce down to a single hypercube.
- Suppose there are exactly two odd dimensional hypercubes. If $r = 2$, then we may conclude by applying

$$2 \text{ good odd} \rightarrow 1 \text{ even (Proposition 7.4),}$$

  if $r = 3$, then we may conclude by applying

$$2 \text{ good odd, } 1 \text{ good even} \rightarrow 1 \text{ even (Proposition 9.2),}$$

  and if $r = 4$ we may conclude by applying in order

$$1 \text{ good odd, } 1 \text{ good even} \rightarrow 1 \text{ odd (Proposition 7.4),}$$
$$1 \text{ good even, } 1 \text{ good odd, } 1 \text{ odd} \rightarrow 1 \text{ even (Proposition 9.2).}$$

  If $r \geq 5$, then there are at least three even dimensional hypercubes, and applying one of

$$2 \text{ good even, } 1 \text{ odd} \rightarrow 1 \text{ odd (Proposition 9.2) or}$$
$$3 \text{ good even, } 1 \text{ odd} \rightarrow 1 \text{ odd (Proposition 9.5),}$$

  we reduce to the case

$$\text{odd number of good even, } 1 \text{ good odd, } 1 \text{ odd.}$$

  We then conclude by repeatedly applying

$$2 \text{ good even, } 1 \text{ odd} \rightarrow 1 \text{ odd (Proposition 9.2)}$$

  until we have one even dimensional hypercube left, after which we apply

$$1 \text{ good even, } 1 \text{ good odd, } 1 \text{ odd} \rightarrow 1 \text{ even (Proposition 9.2).}$$

Hence, we may assume there are at least three odd dimensional hypercubes, with two of these good. By repeatedly applying

2 good even, 1 odd $\to$ 1 odd (Proposition 9.2)

as many times as possible, always avoiding using the good odd dimensional hypercubes, we may assume that there is at most one even dimensional hypercube.

By Theorem 8.1, every product cuboid has a proper decomposition except possibly if one of the chains used in the product cuboid has length 1, or all but one chain is length 2 and the remaining chain is maximal. We note that since we have at most one even dimensional hypercube, there is at most one factor of length 1 in any product cuboid.

If a product cuboid has a factor of length 1, that factor must be from the even dimensional hypercube, and by the goodness of the factor, any two such product cuboids will be disjoint as by definition the length 1 chains are disjoint between members of a family of good almost orthogonal symmetric chain decompositions of an even dimensional hypercube.

We now show that the intersection of a cuboid with a factor of length 1 and a cuboid where exactly one factor is maximal and the remaining factors are of length 2 intersect in at most one element, so chains from the first cuboid automatically intersect chains from the second cuboid almost orthogonally. Indeed, the length 1 factor from the first cuboid necessarily coincides with the maximal factor from the second cuboid (as having a length 2 factor implies the corresponding hypercube is odd dimensional), so for each $1 \leq i \leq r$ the ith factor of the first cuboid and the ith factor of the second cuboid are almost orthogonal chains and are not both maximal. Thus the intersection is of size at most one in each factor, so the two cuboids intersect in at most one element.

Hence we may decompose any cuboids with a factor of length 1 with arbitrary symmetric chain decompositions without worrying about non-almost orthogonal intersections occurring, and we are now left with cuboids where all but one factor is of length 2 and the remaining factor is maximal. Two such cuboids where the maximal chain occurs in different factors intersect in at most one element by almost orthogonality of the chains in the corresponding factors, so it suffices to consider intersections between cuboids where the maximal chain occurs in the same factor. But then we may conclude by Proposition 9.1 that since there are at least two good odd hypercubes, one of the factors containing a 2 must be good. □

Proof of Theorem 3.3. Apply Theorem 3.5 to the good collections of three almost orthogonal symmetric chain decompositions of Q_5 and Q_7 from section 4 to decompose $(Q_5)^a \times (Q_7)^b \cong Q_{5a+7b}$. □

Proof of Corollary 3.4. Given a collection of orthogonal decompositions of Q_{2k-1}, we can construct a collection of orthogonal decomposition for Q_{2k} by, for each decomposition, duplicating each chain and adding the element $2k$ into each element of the duplicate chain as $\binom{2k}{k} = 2\binom{2k-1}{k-1}$. We leave $n = 4$ to the reader (or see [10]). □

11. Proof of Theorem 3.6. In this section, we prove Theorem 3.6 using the main result from [2]. The considerations from the beginning of section 9 still apply.

Proof of Theorem 3.6. By Remark 8.2, we can always apply Theorem 8.1. Applying

2 good even, 1 odd $\to$ 1 odd (Proposition 9.2)

as many times as possible, always avoiding using the good odd dimensional hypercubes, we may assume that there is at most one even dimensional hypercube.

The proof of Theorem 3.5 from section 10 after the final application of Proposition 9.2 now applies verbatim until the very end, where we are left considering intersections between $2 \times 2 \times \cdots \times 2 \times k$ cuboids with at least five 2's and $k \geq 5$ and k in the same factor. But now we instead apply one of the main results of [2] that such cuboids have universally very proper decompositions (called decompositions with no "taut" chains in [2]). ☐

12. Future directions of research. We note that there are two immediate directions of future research from this work. The first would be to find $k \geq 4$ almost-orthogonal chain decompositions of other hypercubes and use our results to bootstrap them to arbitrarily large hypercubes (as was done in [1] for $k = 4$; see Remark 1.1).

The other direction is to extend Theorem 8.1 to allow chains of length 3 and 4 to be designated universally proper. I would expect that a complete classification of products of chains with designations that have universally proper decompositions is possible, and in fact I anticipate the hardest cases were already handled in [2], though I have not tried to carry out the remaining analysis. Knowing this would be very useful in extending the results in this paper to their greatest possible generality to other posets (since, as remarked in the introduction, we only ever used the lengths of symmetric chains of Q_n in our analyses).

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