

MODIFIED DIAGONALS AND LINEAR RELATIONS BETWEEN SMALL DIAGONALS

HUNTER SPINK

ABSTRACT. Let (X, pt) be a pointed smooth projective variety. We prove that the vanishings of the modified diagonal cycles of Gross and Schoen govern the \mathbb{Z} -linear relations between small m -diagonals $\text{pt}^{\{1, \dots, n\} \setminus A} \times \Delta_A$ in the rational Chow ring of X^n for A ranging over m -element subsets of $\{1, \dots, n\}$.

The combinatorial heart of this paper, which may be of independent interest, is showing the \mathbb{Z} -linear relations between elementary symmetric polynomials $e_k(x_{a_1}, \dots, x_{a_m}) \in \mathbb{Z}[x_1, \dots, x_n]$ are generated by the S_n -translates of a certain alternating sum over the facets of a hyperoctahedron.

1. INTRODUCTION

Let (X, pt) be a pointed smooth connected projective variety and $n \in \mathbb{N}$. We denote by $\Delta_m \subset X^m$ for the diagonal, and for any $A \subset \{1, \dots, n\}$ with $|A| = m$, the *small diagonal associated to A* is defined to be

$$\Delta_m(A) := \text{pt}^{\{1, \dots, n\} \setminus A} \times \Delta_m \subset X^{\{1, \dots, n\} \setminus A} \times X^A = X^n,$$

the locus of $(x_1, \dots, x_n) \in X^n$ where $x_i = \text{pt}$ if $i \notin A$ and $x_i = x_j$ if $i, j \in A$.

For example, when $m = 2$ and $n = 3$ we have the three small diagonals

$$\Delta_2(\{1, 2\}) = \{(x, x, \text{pt}) \mid x \in X\} \subset X^3,$$

$$\Delta_2(\{1, 3\}) = \{(x, \text{pt}, x) \mid x \in X\} \subset X^3,$$

$$\Delta_2(\{2, 3\}) = \{(\text{pt}, x, x) \mid x \in X\} \subset X^3.$$

We denote by $Z^\bullet(X^k)$ for the free abelian group of *algebraic cycles*, generated by subvarieties of X^k , graded by codimension. The *modified diagonal cycles* of Gross and Schoen [GS95] are defined as the formal linear combination of small diagonals

$$\Delta'_k := \sum_{\emptyset \neq C \subset \{1, \dots, k\}} (-1)^{k-|C|} \Delta_C(C) \in Z^{(k-1)\dim(X)}(X^k).$$

For example when $k = 3$, the modified diagonal cycle in $Z^{2\dim(X)}(X^3)$ is

$$\Delta'_3 := \Delta_3 - \Delta_2(\{1, 2\}) - \Delta_2(\{1, 3\}) - \Delta_2(\{2, 3\}) + \Delta_1(\{1\}) + \Delta_1(\{2\}) + \Delta_1(\{3\}).$$

The vanishing of Δ'_k in the Chow ring $A^\bullet(X^k)$ (the quotient of $Z^\bullet(X^k)$ by rational equivalence of cycles) has been intensely studied in the context of diagonal decompositions in Chow groups. We note that the class in the Chow ring is sensitive to the choice of pt (unlike the analogous situation in cohomology).

In this paper we give a new interpretation of the vanishings of the modified diagonal cycles of Gross and Schoen [GS95] in the rational Chow ring $A^\bullet(X^n)_\mathbb{Q} := A^\bullet(X^n) \otimes \mathbb{Q}$.

Affiliation: Harvard University. Email: hspink@math.harvard.edu.

Let $\mathbb{Z}^{\binom{n}{m}}$ denote the free abelian group on elements $[A]$ with $A \subset \{1, \dots, n\}$ a subset of size m . We define the map of abelian groups

$$\begin{aligned} \Phi_{m,n,X} : \mathbb{Z}^{\binom{n}{m}} &\rightarrow A^\bullet(X^n)_{\mathbb{Q}} \\ [A] &\mapsto [\Delta_m(A)], \end{aligned}$$

and denote by

$$R(m, n, X) = \ker \Phi_{m,n,X}$$

the group of \mathbb{Z} -linear relations between the classes $[\Delta_m(A)] \in A^\bullet(X^n)_{\mathbb{Q}}$ as A ranges over m -element subsets of $\{1, \dots, n\}$. Elements of $R(m, n, X)$ are called *relations*, and we write $\sum a_A [A] \sim \sum b_A [A]$ to indicate $\sum (a_A - b_A) [A] \in R(m, n, X)$.

Theorem 1.1. Let $k \in [2, \infty]$ be the smallest number such that $[\Delta'_k] \in A^\bullet(X^k)_{\mathbb{Q}}$ vanishes (∞ if no such number exists). Then the group of \mathbb{Z} -linear relations $R(m, n, X)$ is equal to a subgroup $G_k(m, n) \subset \mathbb{Z}^{\binom{n}{m}}$ depending only on k, m, n . This group satisfies $G_k(m, n) = 0$ for $k \geq m + 1$, and

$$0 = G_{m+1}(m, n) \subsetneq G_m(m, n) \subsetneq \dots \subsetneq G_2(m, n) \subsetneq \mathbb{Z}^{\binom{n}{m}}.$$

For $2 \leq k \leq m$ there is a single distinguished relation involving generators A with $A \subset \{1, \dots, m+k\}$ independent of n (the “ k -hyperoctahedral m -relation” from Definition 2.3), and $G_k(m, n)$ is the group generated by the S_n -translates of this relation.

O’Grady [O’G14] showed that $\Delta'_k = 0 \implies \Delta'_\ell = 0$ for all $\ell \geq k$, so the following is an immediate corollary.

Corollary 1.2. If $\Delta'_\ell = 0$ then $G_\ell(m, n) \subset R(m, n, X)$.

We have the following vanishing results of modified diagonals from the literature.

- In [GS95], Gross and Schoen showed that
 - $\Delta'_k = 0$ precisely if $k \geq 2$ for $X = \mathbb{P}^1$,
 - $\Delta'_k = 0$ precisely if $k \geq 3$ for X of genus 1, and
 - $\Delta'_3 = 0$ for X a hyperelliptic curve with $\text{pt} \in X$ a Weierstrass point.
- In [BV04], Beauville and Voisin showed that
 - $\Delta'_3 = 0$ on a $K3$ -surface X if $\text{pt} \in X$ lies on a rational curve.
- In [Voi15], Voisin showed that
 - if X is a smooth projective connected variety of dimension n swept out by irreducible curves of genus g supporting a zero-cycle rationally equivalent to $\text{pt} \in X$, then we have $\Delta'_m = 0$ for $m \geq (n+1)(g+1)$.
- In [MY16], Moonen and Yin showed that
 - $\Delta'_n = 0$ on a g -dimensional abelian variety precisely when $n \geq 2g+1$, and
 - $\Delta'_n = 0$ on a curve of genus g whenever $n \geq g+2$ (which is sharp for a generic pointed curve, see [Qiz14]).

As $\Delta'_1 = X \neq 0$ and it is known (and easy to show) that only for $X = \text{pt}$ or \mathbb{P}^1 do we have $\Delta'_2 = 0$, we deduce the following as a corollary of Theorem 1.1.

Corollary 1.3. Denote by $G_k(m, n)$ the group from Theorem 1.1 (c.f. Definition 2.3). Then for the various smooth pointed projective varieties (X, pt) below, the \mathbb{Z} -linear relations between the classes of the $\binom{n}{m}$ small diagonals $[\Delta_m(A)] \in A^\bullet(X^n)_{\mathbb{Q}}$ with $A \subset \{1, \dots, n\}$ of size m are given as follows.

- $G_2(m, n)$ if $X = \mathbb{P}^1$ and $G_3(m, n)$ is X is of genus 1.
- $G_{2g+1}(m, n)$ if X is a g -dimensional abelian variety.
- $G_{g+2}(m, n)$ if X is a generic pointed curve of genus g .
- $G_3(m, n)$ if X is a K_3 surface and $\text{pt} \in X$ lying on a rational curve.
- $G_3(m, n)$ if X is a hyperelliptic curve with $\text{pt} \in X$ a Weierstrass point.

In the remaining cases where we do not necessarily know the minimal k such that $\Delta'_k = 0$, Corollary 1.2 yields a partial list of relations between the classes $[\Delta_m(A)] \in A^\bullet(X^n)_\mathbb{Q}$.

Example 1.4. Suppose $X = \mathbb{P}^1$, $m = 2$, and $n = 4$. Then

$$A^\bullet((\mathbb{P}^1)^4)_\mathbb{Q} := \mathbb{Q}[H_1, H_2, H_3, H_4]/(H_1^2, H_2^2, H_3^2, H_4^2)$$

where H_i is the hyperplane class in \mathbb{P}^1 pulled back to $(\mathbb{P}^1)^4$ under the i th projection, and $\Phi_{2,4,\mathbb{P}^1}$ takes $[\{i, j\}] \mapsto \Delta_2(\{i, j\}) = H_i + H_j$.

Because $\Delta'_2 = 0$, $R(2, 4, \mathbb{P}^1)$ is generated by the relations

$$[\{1, 2\}] + [\{3, 4\}] \sim [\{1, 3\}] + [\{2, 4\}] \sim [\{1, 4\}] + [\{2, 3\}].$$

These are the WDVV relations, which arise from considering the subvariety C_λ in $(\mathbb{P}^1)^4$ of collinear (x_1, x_2, x_3, x_4) with j -invariant λ , and degenerating λ to $0, 1, \infty$. The relation $[\{1, 2\}] - [\{2, 3\}] + [\{3, 4\}] - [\{4, 1\}]$ is in fact a 2-hyperoctahedral 2-relation, and the above theorem shows the relations in $A^\bullet_\mathbb{Q}((\mathbb{P}^1)^n)$ between the classes of $\Delta_2(\{i, j\})$ are generated by the relations $[\{i, j\}] + [\{k, l\}] \sim [\{i, k\}] + [\{j, l\}]$, which are the pullbacks of the WDVV relations under the various projections $(\mathbb{P}^1)^n \rightarrow (\mathbb{P}^1)^4$.

The combinatorial heart of this paper Theorem 2.4 shows that certain “ $(k + 1)$ -hyperoctahedral m -relations” (see Definition 2.3) govern the \mathbb{Z} -linear relations between elementary symmetric polynomials $e_k(x_{a_1}, \dots, x_{a_m}) \in \mathbb{Z}[x_1, \dots, x_n]$ where $\{a_1, \dots, a_m\} \subset \{1, \dots, n\}$ ranges over m -element subsets, which the author believes is an interesting result in its own right.

The geometric heart of this paper is a Chow motive computation, which allows us to extract useful information about classes in $A^\bullet(X^n)_\mathbb{Q}$ despite having essentially no information about the ring itself (in particular no Künneth type decomposition as in Example 1.4). We will decompose the diagonal class in $A^\bullet((X^n)^2)_\mathbb{Q}$ in such a way that convolving with the pieces yields a system of orthogonal idempotent endomorphisms of $A^\bullet(X^n)_\mathbb{Q}$, which consequently decomposes $A^\bullet(X^n)_\mathbb{Q}$ into the direct sum of the images of the idempotents. Our key insight is that we can produce such a decomposition where the non-zero components of the diagonal class in these summands govern the \mathbb{Z} -linear relations between the small diagonals in $A^\bullet(X^n)_\mathbb{Q}$.

The structure of this paper is as follows. In Section 2 we describe the “hyperoctahedral relations” and state our main combinatorial result Theorem 2.4 classifying the \mathbb{Z} -linear relations between polynomials $e_k(x_{a_1}, \dots, x_{a_m}) \in \mathbb{Z}[x_1, \dots, x_n]$ where $\{a_1, \dots, a_m\} \subset \{1, \dots, n\}$ ranges over m element subsets. In Section 3 we describe generalities on Chow motives. In Section 4, we prove Theorem 1.1 (in fact a slight generalization involving arbitrary symmetric classes in $A^\bullet(X^n)_\mathbb{Q}$) assuming Theorem 2.4. Finally, in Section 5 we prove Theorem 2.4

2. HYPEROCTAHEDRAL RELATIONS AND SYMMETRIC POLYNOMIALS

We start with the following definitions.

Definition 2.1. Denote by $e_k(x_1, \dots, x_m) \in \mathbb{Z}[x_1, \dots, x_m]$ the k th elementary symmetric polynomial. For a symmetric polynomial $f \in \mathbb{Z}[x_1, \dots, x_m]^{S_m}$ and $A = \{a_1, \dots, a_m\} \subset \{1, \dots, n\}$ an m -element set, we denote by

$$f(A) := f(x_{a_1}, \dots, x_{a_m}) \in \mathbb{Z}[x_1, \dots, x_n].$$

We define the map of abelian groups

$$\begin{aligned} \Phi_{m,n,f} &:= \mathbb{Z}^{\binom{n}{m}} \rightarrow \mathbb{Z}[x_1, \dots, x_n] \\ [A] &\mapsto f(A), \end{aligned}$$

and let

$$R(m, n, f) := \ker \Phi_{m,n,f}$$

denote the group of \mathbb{Z} -linear relations between the polynomials $f(A)$. Elements of $R(m, n, X)$ are called *relations*, and we write $\sum a_A[A] \sim \sum b_A[A]$ to indicate $\sum (a_A - b_A)[A] \in R(m, n, f)$.

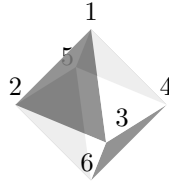
Note the strong resemblance of these definitions to the map $\Phi_{m,n,X}$ and group $R(m, n, X)$ from the introduction. To motivate the hyperoctahedral relations in Definition 2.3, we consider the following example.

Example 2.2. Let $m = 2$, $n = 3$, and $f(x_1, x_2, x_3) = e_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$. Then we can view a relation $\sum_{|A|=3} \lambda_A[A]$, which we recall means

$$\sum_{|A|=3} \lambda_A e_2(A) = 0,$$

as describing a formal \mathbb{Z} -linear combination of triangles $\{i, j, k\} \subset \{1, \dots, n\}$ such that after replacing each $\{i, j, k\}$ with the sum of its three edges $\{i, j\} + \{j, k\} + \{i, k\}$, the sum becomes zero.

If we have a triangulation of a surface such that the faces can be alternately colored black and white, then we can alternately sum the triangles on the surface to get such a relation. The smallest non-trivial instance of this occurs for an octahedron. The sum of $\{i, j, k\}$ over all dark triangles minus the sum over light triangles in the octahedron



is what we will in Definition 2.3 call a 3-hyperoctahedral 3-sum, yielding the relation

$$\begin{aligned} &[\{1, 2, 3\}] + [\{1, 4, 5\}] + [\{6, 3, 4\}] + [\{6, 2, 5\}] \\ &\sim [\{1, 3, 4\}] + [\{1, 2, 5\}] + [\{6, 2, 3\}] + [\{6, 4, 5\}]. \end{aligned}$$

Theorem 2.4 specialized to this case shows that such relations generate all \mathbb{Z} -linear relations between the $e_2(A)$ with $|A| = 3$.

Definition 2.3. Given a set $B \subset \{1, \dots, n\}$ of size $m - k$ and ordered pairs $C_i = (c_{i,0}, c_{i,1})$ from $\{1, \dots, n\} \setminus B$ for $i = 1, \dots, k$ with all $c_{i,j}$ distinct, we say that the element

$$\sum_{(\epsilon_1, \dots, \epsilon_k) \in \{0,1\}^k} (-1)^{\sum_{i=1}^k \epsilon_i} [B \sqcup \{c_{1,\epsilon_1}, \dots, c_{k,\epsilon_k}\}]$$

is a k -hyperoctahedral m -sum in $\mathbb{Z}^{\binom{n}{m}}$. Define $G_k(m, n) \subset \mathbb{Z}^{\binom{n}{m}}$ to be the subgroup generated by k -hyperoctahedral m -sums.

The parameter k may be thought of as the “dimension” of the hyperoctahedron, and when $k < m$ this expression can be thought of as an $(m - k)$ iterated cone with apex(es) B over a k -hyperoctahedral k -sum, the alternating sum over facets of a k -dimensional hyperoctahedron as in Example 2.2 in the case $k = 3$ with $C_1 = (1, 6)$, $C_2 = (2, 4)$, $C_3 = (3, 5)$.

The following theorem, whose proof we defer to Section 5, is our central combinatorial result.

Theorem 2.4. Let $k \leq m \leq n$ be integers. Then $R(m, n, e_k) = G_{k+1}(m, n) \subset \mathbb{Z}^{\binom{n}{m}}$, i.e. the \mathbb{Z} -linear relations between the $e_k(A)$ are generated by the $(k + 1)$ -hyperoctahedral m -sums.

Furthermore, we have the sequence of inclusions

$$0 \subsetneq G_m(m, n) \subsetneq \dots \subsetneq G_0(m, n) = \mathbb{Z}^{\binom{n}{m}}.$$

Corollary 2.5. If $f \in \mathbb{Z}[x_1, \dots, x_m]^{S_m}$ is a symmetric polynomial, then $R(m, n, f) = G_{k+1}(m, n)$, where k is the largest number of distinct x_i to appear in a non-zero monomial.

Proof of Corollary 2.5. For $B = (b_1, \dots, b_j)$ with $b_1 \leq b_2 \leq \dots \leq b_j$, if we let g_B be the sum of all distinct monomials $x_{i_1}^{b_1} \cdots x_{i_j}^{b_j}$ with $\{i_1, \dots, i_j\} \subset \{1, \dots, m\}$, then we can write $f = \sum_B \lambda_B g_B$ for some coefficients λ_B , and as distinct monomials are linearly independent,

$$R(m, n, f) = \bigcap_{\lambda_B \neq 0} R(m, n, g_B).$$

But it is easy to see that the relations between $g_B(A)$ are identical to the relations between $e_j(A)$, which by Theorem 2.4 is given by $G_{j+1}(m, n)$. As the $G_i(m, n)$ are nested, the result follows. \square

3. GENERALITIES ON CHOW MOTIVES

We now recall general facts about Chow motives that we will need. The Chow ring $A^\bullet(X)$ does not satisfy a Künneth formula identifying $A^\bullet(X^n)$ with $A^\bullet(X)^{\otimes n}$, even after tensoring with \mathbb{Q} . We still have a map $A^\bullet(X)_{\mathbb{Q}}^{\otimes n} \rightarrow A^\bullet(X^n)_{\mathbb{Q}}$, but it is potentially non-injective and non-surjective, and in general there is no simple description of $A^\bullet(X^n)_{\mathbb{Q}}$. By using Chow motives we will see that certain correspondences provide a type of substitute for the Künneth formula, which will be sufficient for our purposes.

We say that a correspondence on X is an element of $A^\bullet(X^2)_{\mathbb{Q}}$. Let Γ be a correspondence on X , which induces an endomorphism of $A^\bullet(X)_{\mathbb{Q}}$ via

$$\Gamma : \alpha \mapsto (\pi_2)_*((\pi_1^* \alpha) \cap \Gamma)$$

where π_i is the projection $X^2 \rightarrow X$ onto the i th factor. For any variety S , Γ similarly induces an endomorphism of $A^\bullet(S \times X)_{\mathbb{Q}}$. There is a notion of composition of correspondences, which for $\Gamma_1, \Gamma_2 \in A^\bullet(X^2)_{\mathbb{Q}}$ is defined by

$$\Gamma_2 \circ \Gamma_1 = (\pi_{13})_*(\pi_{12}^* \Gamma_1 \cup \pi_{23}^* \Gamma_2)$$

where π_{ij} is the projection $X^3 \rightarrow X^2$ onto the i, j factors. On the level of functions, $\Gamma_2 \circ \Gamma_1$ induces the composite endomorphism of $A^\bullet(S \times X)_\mathbb{Q}$. Suppose that Γ is *idempotent*, which means that

$$\Gamma = \Gamma \circ \Gamma.$$

In particular, $\Gamma \in A^{\dim(X)}(X^2)_\mathbb{Q}$ so the associated function doesn't shift the grading. An effective Chow motive is a pair of the form (X, Γ) with Γ an idempotent correspondence on X . The identity for composition is the idempotent $\Gamma = \Delta_2$, whose associated endomorphism on $A^\bullet(S \times X)_\mathbb{Q}$ is the identity.

Example 3.1. Let $\Gamma = [X \times \text{pt}]$. Then

$$\Gamma \circ \Gamma = (\pi_{13})_*([X \times \text{pt} \times X] \cap [X \times X \times \text{pt}]) = (\pi_{13})_*([X \times \text{pt} \times \text{pt}]) = \Gamma,$$

so Γ is idempotent, and (X, Γ) is an effective Chow motive. This Chow motive was used previously by Moonen and Yin [MY16] to study modified diagonal cycles.

More generally, there is a notion of homomorphism and tensor product for Chow motives, and if $\gamma \in A^k(X)_\mathbb{Q}$, has a Poincaré dual γ^* with respect to the degree pairing, then $(X, \gamma^* \boxtimes \gamma)$ is isomorphic to the motive $\mathbb{L}^{\otimes k}$ where \mathbb{L} is the *Lefschetz motive* $(\mathbb{P}^1, \mathbb{P}^1 \times \{\text{pt}\})$, see [Kim05].

As Γ is idempotent and Δ_2 corresponds to the identity, $\Gamma, \Delta_2 - \Gamma$ are orthogonal idempotents, so the images of their associated functions $A^\bullet(X)_\mathbb{Q} \rightarrow A^\bullet(X)_\mathbb{Q}$ direct sum to $A^\bullet(X)_\mathbb{Q}$. Given classes $Y_i \in A^\bullet(X_i)_\mathbb{Q}$, denote by $Y_1 \boxtimes Y_2 := \pi_1^* Z_1 \cap \pi_2^* Z_2 \in A^\bullet(X_1 \times X_2)_\mathbb{Q}$. The 2^n elements $\Gamma^{\boxtimes \{1, \dots, k\} \setminus B} \boxtimes (\Delta_2 - \Gamma)^{\boxtimes B}$ for $B \subset \{1, \dots, n\}$ form an orthogonal system of idempotent correspondences of X^n in $A^\bullet((X^2)^n)_\mathbb{Q} = A^\bullet((X^n)^2)_\mathbb{Q}$ summing to the identity correspondence $\Delta_2^{\boxtimes \{1, \dots, n\}}$, so

$$A^\bullet(X^n)_\mathbb{Q} = \bigoplus_{B \subset \{1, \dots, n\}} \text{Im}(\Gamma^{\boxtimes \{1, \dots, n\} \setminus B} \boxtimes (\Delta_2 - \Gamma)^{\boxtimes B}).$$

We remark that on for any product $X_1 \times X_2$ with correspondences $\Gamma_i \in A^\bullet((X_i)^2)_\mathbb{Q}$ we have $\Gamma_1 \boxtimes \Gamma_2 = (\Gamma_1 \boxtimes (\Delta_2)_{X_2}) \circ ((\Delta_2)_{X_1} \boxtimes \Gamma_2)$. Hence each of the above correspondences is the composition of n commuting correspondences on X^n , with the i th correspondence inducing the endomorphism of $A^\bullet(X^{i-1} \times X \times X^{n-i})_\mathbb{Q}$ from the associated correspondence on X .

Note that Im does not distribute over \boxtimes in general because there may be classes in $A^\bullet(X^n)_\mathbb{Q}$ which are not in the image of $A^\bullet(X)_\mathbb{Q}^{\otimes m} \rightarrow A^\bullet(X^m)_\mathbb{Q}$.

4. PROOF OF THEOREM 1.1

We now prove Theorem 1.1. To do this, we prove a more general result involving a modified-diagonal type construction applied to general $\alpha \in A^\bullet(X^n)_{\mathbb{Q}}^{S_n}$.

Definition 4.1. For subsets $A, B \subset \{1, \dots, n\}$ with $|A| = m$, $|B| = k$, and an element $\alpha \in A^\bullet(X^n)_{\mathbb{Q}}^{S_n}$, we denote by

$$\alpha_m(A) = (\Gamma^{\boxtimes \{1, \dots, n\} \setminus A} \boxtimes \Delta_2^{\boxtimes A})(\alpha),$$

and

$$\alpha'_k(B) = (\Gamma^{\boxtimes \{1, \dots, n\} \setminus B} \boxtimes (\Delta_2 - \Gamma)^{\boxtimes B})(\alpha) = \sum_{C \subset B} (-1)^{k-|C|} \alpha_{|C|}(C)$$

Remark 4.2. For applications to modified diagonals, we will be taking $\alpha = [\Delta_n] \in A^\bullet(X^n)$ the diagonal class and $\Gamma = [X \times \text{pt}]$ as in Example 3.1, but we remark that we have not defined classes α_m and α'_k for arbitrary $\alpha \in A^\bullet(X^n)_{\mathbb{Q}}^{S_n}$. We will check at the start of the proof of Theorem 1.1 that these definitions are consistent with the definitions of diagonal and modified diagonal cycles from the introduction.

Theorem 4.3. Let $\alpha \in A^\bullet(X^n)_{\mathbb{Q}}^{S_n}$, and let S be the set of ℓ such that $\alpha'_\ell(B) \neq 0$ for one (or equivalently all) B with $|B| = \ell$. Then the kernel of the map of groups

$$\begin{aligned} \Phi_{m,n,\alpha,X} : \mathbb{Z}^{\binom{n}{m}} &\rightarrow A^\bullet(X^n)_{\mathbb{Q}} \\ [A] &\mapsto \alpha_m(A) \end{aligned}$$

is $G_{\max S+1}(m, n)$.

Proof assuming Theorem 2.4. Note that by writing $\Delta_2 = \Gamma + (\Delta_2 - \Gamma)$, we can expand out $\alpha_m(A)$ in terms of $\alpha'_\ell(B)$ classes, and obtain the direct summand decomposition

$$\alpha_m(A) = \sum_{\ell \in S} \sum_{B \subset A, |B|=\ell} \alpha'_\ell(B) \subset \bigoplus_{\ell \in S} \bigoplus_{B \subset A, |B|=\ell} \text{Im}(\Gamma^{\boxtimes \{1, \dots, n\} \setminus B} \boxtimes (\Delta_2 - \Gamma)^{\boxtimes B}).$$

In fact, we note that the sum lies in the free abelian subgroup

$$\bigoplus_{\ell \in S} \bigoplus_{B \subset \{1, \dots, n\}, |B|=\ell} \mathbb{Z} \alpha'_\ell(B)$$

(the freeness follows because there is no torsion as we are working in the rational Chow ring, and there are no relations between different summands). Identifying the generators $\alpha'_\ell(B)$ with the squarefree monomials $\prod_{i \in B} x_i$, we see that $\alpha_m(A)$ corresponds to the polynomial $(\sum_{i \in S} e_i)(A)$, so we thus have the kernel is $R(m, n, (\sum_{i \in S} e_i)(A))$, and we conclude by Corollary 2.5. \square

Proof of Theorem 1.1 assuming Theorem 2.4. With $\alpha = [\Delta_n] \in A^\bullet(X^n)$ the diagonal class and $\Gamma = [X \times \text{pt}]$ as in Example 3.1, we will first check that Definition 4.1 is consistent with our definitions of diagonal cycles and modified diagonal cycles from the introduction. Indeed, with the definition from Definition 4.1,

$$\Delta_m(A) = (\Gamma^{\boxtimes \{1, \dots, n\} \setminus A} \boxtimes \Delta_2^{\boxtimes A})(\Delta_n) = (\pi_2)_*(Z_A \cap (\Delta_n \times X^n))$$

where Z_A is given by

$$X^{\{1, \dots, n\} \setminus A} \times \Delta_2^A \times \text{pt}^{\{1, \dots, n\} \setminus A} \subset X^{\{1, \dots, n\} \setminus A} \times X^A \times X^A \times X^{\{1, \dots, n\} \setminus A} = X^n \times X^n$$

Then $Z_A \cap (\Delta_n \times X^n)$ is the product of the diagonal $\Delta_{n+|A|} \subset X^{\{1, \dots, n\} \setminus A} \times X^A \times X^A$ with $\text{pt}^{\{1, \dots, n\} \setminus A} \in X^{\{1, \dots, n\} \setminus A}$, and we see that $\Delta_m(A) = (\pi_2)_*(Z_A \cap (\Delta_n \times X^n))$ is thus precisely the same as it was defined in the introduction. Also, the inclusion-exclusion from Definition 4.1 implies for any set B with size ℓ ,

$$\Delta'_\ell(B) = \text{pt}^{\{1, \dots, n\} \setminus B} \times \Delta'_\ell \subset X^{\{1, \dots, n\} \setminus B} \times X^B = X^n.$$

The class Δ'_ℓ is the pushforward of $\Delta'_\ell(B)$ along the projection $X^n \rightarrow X^B \cong X^\ell$, so for any B of size ℓ , Δ_ℓ vanishes if and only if $\Delta'_\ell(B)$ vanishes. Applying Theorem 4.3 (recalling by [O'G14] that the set S of ℓ such that Δ'_ℓ does not vanish is precisely $\{1, \dots, k-1\}$), we get $R(m, n, X)$ is equal to the subgroup $G_k(m, n)$ as desired. The nesting and generation by S_n -translates follows from Theorem 2.4. \square

5. PROOF OF THEOREM 2.4

In this section, we prove Theorem 2.4.

Proof of Theorem 2.4. First, we show that the $e_k(A)$ polynomials satisfy every $(k+1)$ -hyperoctahedral m -relation, i.e.

$$\sum_{(\epsilon_1, \dots, \epsilon_{k+1}) \in \{0,1\}^{k+1}} (-1)^{\sum_{i=1}^{k+1} \epsilon_i} e_k(B \sqcup \{c_{1,\epsilon_1}, \dots, c_{k+1,\epsilon_{k+1}}\}),$$

where $|B| = m - (k+1)$, and $C_i = (c_{i,0}, c_{i,1})$ are $k+1$ ordered pairs disjoint from B and from each other. Indeed, fix a monomial $x_{i_1} \dots x_{i_k}$, we will show that its coefficient in the above sum is 0. Because there are $k+1$ disjoint two-element sets C_i , one of the pairs C_ℓ contains none of the indices i_j . Hence, writing the $(k+1)$ -hyperoctahedral m -relation as

$$\sum_{(\epsilon_1, \dots, \widehat{\epsilon_\ell}, \dots, \epsilon_{k+1}) \in \{0,1\}^k} (-1)^{\sum_{i \neq \ell} \epsilon_i} \sum_{r=0}^1 (-1)^r e_k(B \sqcup \{c_{1,\epsilon_1}, \dots, c_{\ell,r}, \dots, c_{k+1,\epsilon_{k+1}}\}),$$

the coefficient of $x_{i_1} \dots x_{i_k}$ vanishes on the inner sum, and hence on the whole relation.

We now show that there are no linear relations between $e_k(A)$ for A ranging over m -element subsets of $\{1, \dots, m+k\}$. Note there are $\binom{m+k}{m}$ polynomials of the form $e_k(A)$ with $A \subset \{1, \dots, m+k\}$ of size m , and they lie in the span of the $\binom{m+k}{k}$ degree k square-free monomials in x_1, \dots, x_{m+k} . Hence, to show linear independence of the $e_k(A)$ it suffices to show that we can write each of these squarefree monomials as a \mathbb{Q} -linear combination of the $e_k(A)$ with $A \subset \{1, \dots, m+k\}$ of size m .

We will do this by inductively showing that all polynomials $x_{i_1} \dots x_{i_\ell} e_{k-\ell}(B)$ with $i_1, \dots, i_\ell \in \{1, \dots, m+k\}$ distinct and $B \subset \{1, \dots, m+k\} \setminus \{i_1, \dots, i_\ell\}$ a subset of size m lie in the \mathbb{Q} -linear span of the $e_k(A)$. This is true for $\ell = 0$. Suppose the result is true for $\ell - 1$, we will show it is true for ℓ . Indeed,

$$(m - (k - \ell))x_{i_1} \dots x_{i_\ell} e_{k-\ell}(B) = x_1 \dots x_{i_{\ell-1}} \sum_{b \in B} e_{k-\ell+1}(\{i_\ell\} \sqcup B \setminus \{b\}) - x_1 \dots x_{i_{\ell-1}} (m - (k - \ell + 1))e_{k-\ell+1}(B).$$

Thus by induction the statement is true for $\ell = k$, and this shows that each of the squarefree monomials is a \mathbb{Q} -linear combination of the $e_k(A)$ as desired.

Now, we prove by induction on k that if $\sum \lambda_A e_k(A) = 0$, then we can write $\sum_A \lambda_A [A]$ as a \mathbb{Z} -linear combination of $(k+1)$ -hyperoctahedral m -sums. For $k = 0$ the result is trivial, so now assume that $k > 0$. If $n \leq m+k$ then the relation must be identically zero by what we have just proved, so assume that $n > m+k$.

Our goal is to first show that we may subtract from $\sum \lambda_A [A]$ a \mathbb{Z} -linear combination of $(k+1)$ -hyperoctahedral m -sums involving elements in $\{1, \dots, n\}$ so that the resulting sum involves only sets in $\{1, \dots, n-1\}$. Write our relation between the $e_k(A)$ as

$$0 = \sum \lambda_A e_k(A) = x_n \sum_{n \in A} \lambda_A e_{k-1}(A \setminus n) + \sum_{n \in A} \lambda_A e_k(A \setminus n) + \sum_{n \notin A} \lambda_A e_k(A),$$

we see that the x_n -coefficient is $\sum_{n \in A} \lambda_A e_{k-1}(A \setminus n)$, so must equal 0. By the induction hypothesis, we know that $\sum_{n \in A} \lambda_A [A \setminus n]$ is the sum of k -hyperoctahedral

$(m-1)$ -relations, so there exists $(m-1-k)$ -element sets $B^i \subset \{1, \dots, n-1\}$ and disjoint pairs $C_j^i = (c_{j,0}^i, c_{j,1}^i) \subset \{1, \dots, n-1\} \setminus B^i$ for $1 \leq j \leq k$ such that

$$\sum_{n \in A} \lambda_A[A \setminus n] = \sum_i \sum_{(\epsilon_1, \dots, \epsilon_k) \in \{0,1\}^k} (-1)^{\sum \epsilon_i} [B^i \sqcup \{c_{1,\epsilon_1}^i, \dots, c_{k,\epsilon_k}^i\}].$$

As $n-1 \geq m+k$ and only $m+k-1$ elements are used in the i th hyperoctahedral sum, there exists an element $r_i \in \{1, \dots, n-1\}$ not used in the i th sum. Then letting $c_{k+1,0}^i = n$ and $c_{k+1,1}^i = r_i$, we have

$$\begin{aligned} \sum_{n \in A} \lambda_A A &= \sum_i \sum_{(\epsilon_1, \dots, \epsilon_{k+1}) \in \{0,1\}^{k+1}} (-1)^{\sum \epsilon_i} [B^i \sqcup \{c_{1,\epsilon_1}^i, \dots, c_{k+1,\epsilon_{k+1}}^i\}] \\ &\quad + \sum_i \sum_{(\epsilon_1, \dots, \epsilon_k) \in \{0,1\}^k} (-1)^{\sum \epsilon_i} [\{r_i\} \sqcup B^i \sqcup \{c_{1,\epsilon_1}^i, \dots, c_{k,\epsilon_k}^i\}], \end{aligned}$$

where the first term on the right hand side is a sum of $(k+1)$ -hyperoctahedral m -sums, and the second term does not involve n . Hence subtracting these $(k+1)$ -hyperoctahedral m -relations from $\sum \lambda_A A$, we obtain a sum involving only sets in $\{1, \dots, n-1\}$.

Since the $e_k(A)$ polynomials satisfy every $(k+1)$ -hyperoctahedral m -relation, subtracting the corresponding sum of $(k+1)$ -hyperoctahedral m -sums from the sum $\sum_A \lambda_A e_k(A)$ still yields a relation between the $e_k(A)$, and we have reduced the problem to one where all of the A lie in $\{1, \dots, n-1\}$. Repeating this we eventually reduce down to where all $A \subset \{1, \dots, m+k\}$. By what we showed earlier the resulting relation between the $e_k(A)$ must in fact be the zero relation. Hence we have written the original relation $0 = \sum_A \lambda_A e_k(A)$ as a \mathbb{Z} -linear combination of $(k+1)$ -hyperoctahedral m -relations as desired, and our induction is complete.

Finally, the nesting of the G_i follows since given a relation between $e_k(A)$, applying the operator $\frac{1}{m-k+1} \sum_{i=1}^n \frac{\partial}{\partial x_i}$ yields the identical relation between $e_{k-1}(A)$. The strictness of the inclusion $G_k(m, n) \subset G_{k+1}(m, n)$ follows as a k -hyperoctahedral m -relation in $G_k(m, n)$ involves $m+k$ elements, so cannot be a relation between the $e_k(A)$ with $A \subset \{1, \dots, m+k\}$ of size m from the linear independence proved earlier, and thus does not lie in $G_{k+1}(m, n)$. \square

Acknowledgements. I would like to thank Dennis Tseng for suggesting to study relations between diagonals and for many fruitful conversations.

REFERENCES

- [BV04] Arnaud Beauville and Claire Voisin. On the Chow ring of a $K3$ surface. *J. Algebraic Geom.*, 13(3):417–426, 2004.
- [GS95] B. H. Gross and C. Schoen. The modified diagonal cycle on the triple product of a pointed curve. *Ann. Inst. Fourier (Grenoble)*, 45(3):649–679, 1995.
- [Kim05] Shun-Ichi Kimura. Chow groups are finite dimensional, in some sense. *Math. Ann.*, 331(1):173–201, 2005.
- [MY16] Ben Moonen and Qizheng Yin. Some remarks on modified diagonals. *Commun. Contemp. Math.*, 18(1):1550009, 16, 2016.
- [O’G14] Kieran G. O’Grady. Computations with modified diagonals. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 25(3):249–274, 2014.
- [Qiz14] Yin Qizheng. *Tautological cycles on curves and Jacobians*. PhD thesis, Radboud University Nijmegen, 2014.
- [Voi15] Claire Voisin. Some new results on modified diagonals. *Geom. Topol.*, 19(6):3307–3343, 2015.