JUDICIOUSLY 3-PARTITIONING 3-UNIFORM HYPERGRAPHS

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ABSTRACT. Bollobás, Reed and Thomason proved every 3-uniform hypergraph $\mathcal{H}$ with m edges has a vertex-partition $V(\mathcal{H}) = V_1 \sqcup V_2 \sqcup V_3$ such that each part meets at least $\frac{1}{3}(1 - \frac{1}{e})m$ edges, later improved to $0.6m$ by Halsegrave and improved asymptotically to $0.65m + o(m)$ by Ma and Yu. We improve this asymptotic bound to $\frac{19}{27}m + o(m)$, which is best possible up to the error term, resolving a special case of a conjecture of Bollobás and Scott.

1. INTRODUCTION

Judicious partitioning problems seek to partition the vertices of a hypergraph $\mathcal{H}$ with m edges such that various quantities are simultaneously maximized. The first such problem was treated by Bollobás and Scott [2], where they proved that one can partition the vertices of a graph with m edges into k parts such that each part contains at most $\frac{m}{k^2} + o(m)$ edges. They later proved [3] that the vertices of a 3-uniform hypergraph can be partitioned into k parts each of which contains at most $\frac{m}{k^3} + o(m)$ edges. Recently Hou, Wu, Zeng and Zhu [6] claimed to have shown the result for 4-uniform hypergraphs with $\frac{m}{k^4} + o(m)$ edges, but their key technical lemma is incorrect.

The judicious partitioning problem we consider involves partitioning the vertices of an r-uniform hypergraph with m edges into k parts so that the minimum number of edges touched by any part is maximized. Bollobás and Scott [4] conjectured that this maximum is $(1 - (1 - \frac{1}{k})^r)m + o(m)$,[1] which if true is optimal up to the error term by considering the complete r-uniform hypergraph. Special cases of this problem have garnered considerable interest, particularly the case $r = k = 3$. Bollobás, Reed and Thomason [1] proved every 3-uniform hypergraph $\mathcal{H}$ has a vertex-partition $V(\mathcal{H}) = V_1 \sqcup V_2 \sqcup V_3$ such that each part meets at least $\frac{1}{3}(1 - \frac{1}{e})m$ edges, which was claimed to have been improved to $\frac{5m-1}{9}$ by Bollobás and Scott [4] (although their proof apparently contains a subtle error, see Haslegrave

[1]We note that this asymptotically strengthens a previously conjectured bound of $\frac{rm}{2r-1}$ for any m when $r = k$ by Bollobás and Thomason (see [1] and [4]).

[5]), improved to $0.6m$ by Halsegrave [5] and improved asymptotically to $0.65m + o(m)$ by Ma and Yu [7].

Here we resolve the conjecture of Bollobás and Scott in this special case.

Theorem 1.1. Every 3-uniform hypergraph $\mathcal{H}$ has a vertex-partition $V(\mathcal{H}) = V_1 \sqcup V_2 \sqcup V_3$ such that each part meets at least $\frac{19}{27}m - O(m^{6/7})$ edges.

Our proof follows a strategy of Bollobás and Scott [3] and Ma and Yu [7]. The key technical lemma in [7] cannot be optimized sufficiently to attain $\frac{19}{27}$, in fact cannot be pushed past $0.7 < \frac{19}{27} = 0.\overline{703}$. We make an additional observation which imposes an additional inequality in this technical lemma.

We now outline the strategy. It would be nice if for a given 3-uniform hypergraph, the uniformly random 3-partition of the vertices worked. Each part would touch in expectation $\frac{19}{27}m + o(m)$ edges, but the presence of high degree vertices prevents these three quantities from simultaneously concentrating around the mean. For example, if there is a pair of vertices which belong to every edge, then a part which contains neither vertex will touch with high probability only $\frac{1}{3}m + o(m)$ edges. To avoid this situation, we partition the vertices according to a two stage process.

We first partition "high" degree vertices $V^{\text{high}} \subseteq V(\mathcal{H})$ into three parts V_1, V_2, V_3, and then take the remaining "low" degree vertices $V^{\text{low}} = V(\mathcal{H}) \setminus V^{\text{high}}$ and assign each of these vertices independently to V_i with some probability p_i with $p_1 + p_2 + p_3 = 1$, so that the expected number of edges touched in each part is at least $\frac{19}{27}m + o(m)$. Because we exclude the high degree vertices from the random process, we have a tight concentration around the mean in each part by the Azuma-Hoeffding inequality (or more specifically the version of the inequality due to McDiarmid [8]), and we conclude.

The threshold for "high" degree vertices and the application of the Azuma-Hoeffding inequality are essentially automatic (if we only desire an $o(m)$ error, we only have to ensure that the threshold for "high" degree is $o(m)$). The challenge is to appropriately partition the "high" degree vertices so that the p_i can be chosen to make each part touch the correct number of edges in expectation.

As in [7], after choosing the high degree threshold, we partition V^{high} as follows. Denote by $E(\mathcal{H})^i$ for $i \in \{0, 1, 2, 3\}$ to be the edges in $E(\mathcal{H})$ which touch i vertices in V^{high} and $3 - i$ vertices in V^{low}. The edges in $E(\mathcal{H})^2$ induce a multi-graph G with vertex set V^{high} obtained by replacing each edge e with $e \cap V^{\text{high}}$. Denote by x_i the number of edges of G within V_i^{high}, and b_{ij} the number of edges of G with endpoints in V_i^{high} and V_j^{high}. We choose our partition $V^{\text{high}} = V_1^{\text{high}} \sqcup V_2^{\text{high}} \sqcup V_3^{\text{high}}$ so that $b_{23} + b_{13} + b_{12}$ is maximal (equivalently $x_1 + x_2 + x_3$ is minimal). The only facts used in [7] about this partition were that $b_{ij} \geq \max(2x_i, 2x_j)$. These inequalites follow by noting that the multigraph between V_i^{high} and V_j^{high} maximizes the number of edges between the two parts for fixed $V_i^{\text{high}} \sqcup V_j^{\text{high}}$.
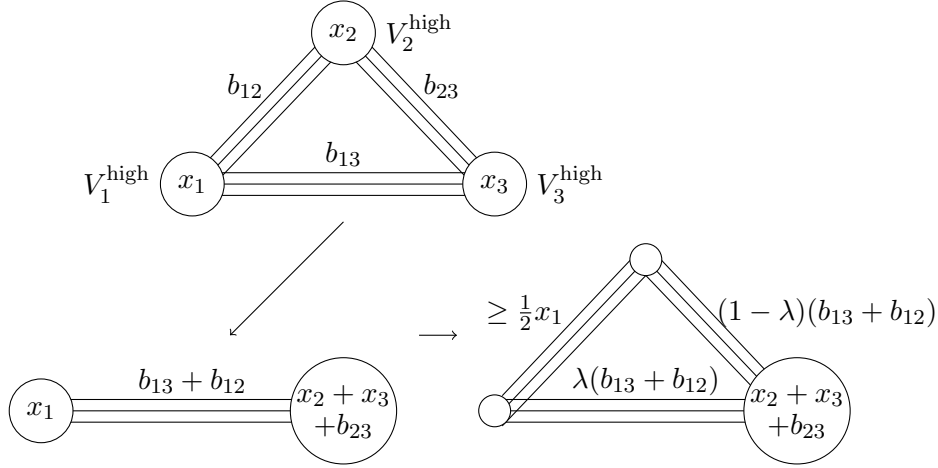
However, these numerical constraints fail to rule out the possibility that G is (close to) a configuration which is complete in V_k^{high}, and complete bipartite between V_k^{high} and the other two parts, which prevents the method from proving a bound exceeding $0.7m$. We identify the additional inequality that if G maximizes $b_{23} + b_{13} + b_{12}$, then $b_{ij} \geq \frac{1}{2}x_k$, where k is the index $\neq i, j$. This arises from considering the effect of moving all vertices from V_i^{high} to V_j^{high}, decreasing $b_{23} + b_{13} + b_{12}$ by b_{ij}, and dividing V_k^{high} between V_k^{high} and V_i^{high} in such a way that b_{ik} is maximized, increasing $b_{23} + b_{13} + b_{12}$ by at least $\frac{1}{2}x_k$.

The core of this paper is a rather technical lemma. This technical lemma is used to show that if we subdivide V^{high} so that $b_{23} + b_{13} + b_{12}$ is maximal, then we can find probabilities p_1, p_2, p_3 as described earlier. Given an instance of $\mathcal{H}$, finding appropriate values of p_i amount to simply solving cubic equations. Hence the laborious method used to prove the inequality is not reflected in the elegance of the final algorithm. We would be very interested in a less opaque derivation of the technical lemma.

The technical lemma is a system of inequalities with 13 variables and 11 degrees of freedom. We ultimately reduce this system to systems of inequalities with 6 variables and 2 degrees of freedom. For the systems which are not tight, we finely subdivide our parameter space and use a computer to check the system on each box. We prove all remaining systems of inequalities in the body of the paper.

2. The technical lemma, its contrapositive, and our strategy

The following technical lemma is the crux of our proof.

Lemma 2.1. For any non-negative variables

$$x_1 + x_2 + x_3 + a_1 + a_2 + a_3 + b_{12} + b_{13} + b_{23} + c = 1$$

with

$$b_{ij} \geq \max(2x_i, 2x_j, \frac{1}{2}x_k),$$

there exists $q_1, q_2, q_3 \in [0, 1]$ with $q_1 + q_2 + q_3 = 2$ such that the following inequalities are satisfied:

$$q_1(b_{23} + x_2 + x_3) + q_1^2(a_2 + a_3) + q_1^3 c \leq \frac{8}{27}$$

$$q_2(b_{13} + x_1 + x_3) + q_2^2(a_1 + a_3) + q_2^3 c \leq \frac{8}{27}$$

$$q_3(b_{12} + x_1 + x_2) + q_3^2(a_1 + a_2) + q_3^3 c \leq \frac{8}{27}.$$

This lemma improves the lemma in [7], which does not impose the inequality $b_{ij} \geq \frac{1}{2}x_k$, and has the constant $\frac{8}{27}$ relaxed to 0.35. The following three sections are devoted to the proof of this lemma.

As the left hand sides of the inequalities in the system are non-decreasing in the q_i and are zero when $q_i = 0$, by symmetry the contrapositive of the lemma is equivalent to the following.

There does not exist $q_1, q_2, q_3 \in [0, 1]$ with $q_1 + q_2 + q_3 = 2$ and non-negative variables

$$x_1 + x_2 + x_3 + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 + c = 1$$

with

$$b_{ij} \geq \max(2x_i, 2x_j, \frac{1}{2}x_k)$$

such that one of the following two systems holds.

System1

$$q_1(b_{23} + x_2 + x_3) + q_1^2(a_2 + a_3) + q_1^3 c > \frac{8}{27}$$

$$q_2(b_{13} + x_1 + x_3) + q_2^2(a_1 + a_3) + q_2^3 c > \frac{8}{27}$$

$$q_3(b_{12} + x_1 + x_2) + q_3^2(a_1 + a_2) + q_3^3 c > \frac{8}{27}.$$

System2 $q_1 = 1$ (so $q_2 + q_3 = 1$) and

$$q_2(b_{13} + x_1 + x_3) + q_2^2(a_1 + a_3) + q_2^3 c > \frac{8}{27}$$

$$q_3(b_{12} + x_1 + x_2) + q_3^2(a_1 + a_2) + q_3^3 c > \frac{8}{27}.$$

We will arrive at a contradiction assuming one of the two systems holds.

We may replace $(a_1, a_2, a_3, c) \mapsto (a_1 + (1 - q_1)c, a_2 + (1 - q_2)c, a_3 + (1 - q_3)c, 0)$ and the corresponding system still holds, so we may assume from now on that $c = 0$. By working with the contrapositive we may assume q_1, q_2, q_3 are fixed and hence we are able to eliminate c in this fashion, which was not exploited previously, but drastically reduces the algebraic complexity of all future systems of inequalities.

Denoting $L = x_1 + x_2 + x_3 + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 + c$ and $L_i = q_i(b_{jk} + x_j + x_k) + q_i^2(x_j + x_k) + q_i^3 c$, our strategy for **System1** is as follows. Keeping the q_i fixed, we repeatedly linearly perturb the remaining variables in such a way that L stays constant and L_1, L_2, L_3 do not decrease. We then perturb in this direction until one of the constraints attains equality. We then merge the associated variables (or set the corresponding variable to 0 if a variable hits zero), and repeat.

By linear algebra, it is clear we can do this (e.g. by choosing any perturbation fixing L, L_1, L_2 and choosing the sign so that L_3 does not decrease), until the number of variables besides q_1, q_2, q_3 is 3. However, we do some manual reductions first to reduce the final number of cases and simplify the exposition.

We handle **System2** in Section 3. **System1** will be dealt with in Sections 4, 5, and 6, completing the proof of Lemma 2.1.

3. System2

Recall we may assume $c = 0$. We may now replace

$$(x_1, x_2, x_3, a_1, a_2, a_3) \mapsto (x_1, 0, 0, 0, 0, 0)$$

$$(b_{23}, b_{13}, b_{12}) \mapsto (\frac{1}{2}x_1, b_{13} + x_3 + q_2(a_1 + a_2 + a_3),$$

$$b_{12} + x_2 + q_3(a_1 + a_2 + a_3) + b_{23} - \frac{1}{2}x_1)$$

so we may assume that $x_2 = x_3 = a_1 = a_2 = a_3 = 0$ and $b_{23} = \frac{1}{2}x_1$. The system is now $b_{12}, b_{13} \geq 2x_1 \geq 0$,

$$b_{12} + b_{13} + \frac{3}{2}x_1 = 1$$

and

$$q_2(b_{13} + x_1) > \frac{8}{27}$$

$$q_3(b_{12} + x_1) > \frac{8}{27}.$$

As $b_{12}, b_{13} \geq 2x_1$, we have $x_1 \leq \frac{2}{11}$. Hence,

$$1 > \frac{8}{27}\left(\frac{1}{b_{13} + x_1} + \frac{1}{b_{12} + x_1}\right)$$

$$\geq \frac{8}{27}\frac{4}{b_{13} + b_{12} + 2x_1}$$

$$= \frac{32}{27}\frac{1}{1 + \frac{1}{2}x_1} \geq \frac{32}{27}\frac{1}{1 + \frac{1}{11}} = \frac{88}{81},$$

where the second inequality is by AM-HM, a contradiction.

4. System1 assuming $x_1 \geq 4x_2$ and $x_2 \geq x_3$

In this section, we prove the result assuming $x_1 \geq 4x_2$ and $x_2 \geq x_3$. The inequalities governing the b_{ij} are thus $b_{12}, b_{13} \geq 2x_1$ and $b_{23} \geq \frac{1}{2}x_1$. Our plan is to vary the variables, with the promise that we never break any of the constraints of **System1**. Recall we may assume $c = 0$. Then we have

$$x_1 + x_2 + x_3 + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(b_{23} + x_2 + x_3) + q_1^2(a_2 + a_3) > \frac{8}{27}$$
$$q_2(b_{13} + x_1 + x_3) + q_2^2(a_1 + a_3) > \frac{8}{27}$$
$$q_3(b_{12} + x_1 + x_2) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

Cases 1a and 1b First, suppose that $x_2 = x_3$. Call this common value x_{23}. Then we have

$$x_1 + 2x_{23} + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(b_{23} + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27}$$
$$q_2(b_{13} + x_1 + x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27}$$
$$q_3(b_{12} + x_1 + x_{23}) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

Now, replace $(x_{23}, b_{23}) \mapsto (x_{23}, b_{23}) + \epsilon(1, -2)$ starting with $\epsilon = 0$, and increase ϵ until either $x_1 = 4x_{23}$ or $b_{23} = \frac{1}{2}x_1$.

Case1a Suppose first that $x_1 = 4x_{23}$, so that $b_{12}, b_{13} \geq 8x_{23}$, $b_{23} \geq 2x_{23}$, and $x_{23} \geq 0$. Then we have

$$6x_{23} + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(b_{23} + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27}$$
$$q_2(b_{13} + 5x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27}$$
$$q_3(b_{12} + 5x_{23}) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

Now, replace $(x_{23}, b_{23}, b_{13}, b_{12}) \mapsto (x_{23}, b_{23}, b_{13}, b_{12}) + \epsilon(1, 2, -4, -4)$ starting with $\epsilon = 0$, and increase ϵ until either $b_{12} = 8x_{23}$ or $b_{13} = 8x_{23}$. Without loss of generality assume $b_{12} = 8x_{23}$. Then we have

System1a $b_{13} \geq 8x_{23}$ and $b_{23} \geq 2x_{23}$ and $x_{23} \geq 0$ and

$$14x_{23} + b_{23} + b_{13} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(b_{23} + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27}$$

$$q_2(b_{13} + 5x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27}$$

$$q_3(13x_{23}) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

Case1b Suppose now instead that $b_{23} = \frac{1}{2}x_1$. Then we have

$$\frac{3}{2}x_1 + 2x_{23} + b_{12} + b_{13} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(\frac{1}{2}x_1 + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27}$$

$$q_2(b_{13} + x_1 + x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27}$$

$$q_3(b_{12} + x_1 + x_{23}) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

Now, replace $(x_1, b_{12}, b_{13}) \mapsto (x_1, b_{12}, b_{13}) + \epsilon(1, -\frac{3}{4}, -\frac{3}{4})$ starting with $\epsilon = 0$, and increase ϵ until either $b_{12} = 2x_1$ or $b_{13} = 2x_1$. Without loss of generality assume $b_{12} = 2x_1$. Then we have

System1b $b_{13} \geq 2x_1 \geq 8x_{23} \geq 0$ and

$$\frac{7}{2}x_1 + 2x_{23} + b_{13} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(\frac{1}{2}x_1 + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27}$$

$$q_2(b_{13} + x_1 + x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27}$$

$$q_3(3x_1 + x_{23}) + q_3(a_1 + a_2) > \frac{8}{27}.$$

Cases 1c and 1d Now, we suppose that $x_2 \geq x_3$ and equality is not attained. We will do similar replacements of variables, but if we ever have $x_2 = x_3$ then we can reduce to either **System1a** or **System1b**, so we may suppose this never happens.

Replace $(x_3, b_{23}) \mapsto (x_3, b_{23}) + \epsilon(1, -1)$ starting with $\epsilon = 0$, and increase ϵ until $b_{23} = \frac{1}{2}x_1$. Now, replace $(x_3, b_{13}) \mapsto (x_3, b_{13}) + \epsilon(1, -1)$ starting with $\epsilon = 0$, and increase ϵ until $b_{13} = 2x_1$. Then we have

$$\frac{7}{2}x_1 + x_2 + x_3 + b_{12} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(\frac{1}{2}x_1 + x_2 + x_3) + q_1^2(a_2 + a_3) > \frac{8}{27}$$
$$q_2(3x_1 + x_3) + q_2^2(a_1 + a_3) > \frac{8}{27}$$
$$q_3(b_{12} + x_1 + x_2) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

Replace $(x_2, b_{12}) \mapsto (x_2, b_{12}) + \epsilon(1, -1)$ starting with $\epsilon = 0$, and increase ϵ until either $b_{12} = 2x_1$ or $x_1 = 4x_2$.

Case1c Suppose first that $b_{12} = 2x_1$. Then we have

System1c $x_1 \geq 4x_2 \geq 4x_3 \geq 0$ and

$$\frac{11}{2}x_1 + x_2 + x_3 + a_1 + a_2 + a_3 = 1$$

and

$$q_1(\frac{1}{2}x_1 + x_2 + x_3) + q_1^2(a_2 + a_3) > \frac{8}{27}$$
$$q_2(3x_1 + x_3) + q_2^2(a_1 + a_3) > \frac{8}{27}$$
$$q_3(3x_1 + x_2) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

Case1d Finally, suppose that $x_1 = 4x_2$. Then we have

System1d $b_{12} \geq 8x_2 \geq 8x_3 \geq 0$ and

$$15x_2 + x_3 + b_{12} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(3x_2 + x_3) + q_1^2(a_2 + a_3) > \frac{8}{27}$$
$$q_2(12x_2 + x_3) + q_2^2(a_1 + a_3) > \frac{8}{27}$$
$$q_3(b_{12} + 5x_2) + q_3^2(a_1 + a_3) > \frac{8}{27}.$$

4.1. **Solving System1a,b,c,d.** For each of these systems, we carry out the procedure described at the end of Section 2. This process terminates when we have 3 variables apart from q_1, q_2, q_3, and for each system there are $\binom{6}{3} = 20$ resulting cases. For each of **System1a,b,c,d** one of the 20 resulting cases will be when $a_1 = a_2 = a_3 = 0$, and one of the resulting cases will be when the remaining 3 variables are equal to zero.

In the former case, when $a_1 = a_2 = a_3 = 0$, we have

$$2 > \frac{8}{27}\left(\frac{1}{b_{23} + x_2 + x_3} + \frac{1}{b_{13} + x_1 + x_3} + \frac{1}{b_{12} + x_1 + x_2}\right)$$
$$\geq \frac{8}{27}\frac{9}{b_{23} + x_2 + x_3 + b_{13} + x_1 + x_3 + b_{12} + x_1 + x_2}$$
$$= \frac{8}{3(1 + x_1 + x_2 + x_3)} \geq \frac{8}{3(1 + \frac{1}{3})} = 2,$$

where the second inequality is by AM-HM, a contradiction.

In the latter case, we have $a_1 + a_2 + a_3 = 1$ and

$$2 > \sqrt{\frac{8}{27}}\left(\frac{1}{\sqrt{a_2 + a_3}} + \frac{1}{\sqrt{a_1 + a_3}} + \frac{1}{\sqrt{a_1 + a_2}}\right).$$

The right hand side is at least 2 by Jensen's inequality applied to $f(x) = \frac{1}{\sqrt{x}}$, a contradiction.

Aside from these two cases for each system, we handle all other cases by computer as follows. Let $\tilde{q}_i$ be the minimum of 1 and the solution to the quadratic equation $L_i = \frac{8}{27}$ (recall that $c = 0$). Then we may equivalently check that $\tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3 \geq 2$ for any choice of the remaining variables. As it turns out, the infimum of $\tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3$ is strictly larger than 2 in all remaining cases, which we can check by subdividing the space of the remaining variables very finely (boxes of dimension $0.001 \times 0.001 \times 0.001$ suffices in all cases), and crudely bounding in each box the smallest possible value of $\tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3$. We report the results of these computations in the Appendix.

5. SYSTEM1 ASSUMING $4x_2 \geq x_1 \geq x_2 \geq x_3$

We assume from now on that $4x_2 \geq x_1 \geq x_2 \geq x_3$, and so the inequalities with the b_{ij} become $b_{12}, b_{13} \geq 2x_1$, $b_{23} \geq 2x_2$. The symmetry of **System1**, along with the previous two sections, completes the verification of the technical lemma. As in the previous section, we will use the method described at the end of Section 2. However, if we ever attain equality in $4x_2 \geq x_1$, then we may stop and use Section 4, so we may assume from now on that we never attain equality in $4x_2 \geq x_1$. Recall we may assume $c = 0$. Then we have

$$x_1 + x_2 + x_3 + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(b_{23} + x_2 + x_3) + q_1^2(a_2 + a_3) > \frac{8}{27}$$
$$q_2(b_{13} + x_1 + x_3) + q_2^2(a_1 + a_3) > \frac{8}{27}$$
$$q_3(b_{12} + x_1 + x_2) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

Replace $(x_1, x_2, x_3, b_{23}, b_{13}, b_{12}) \mapsto (x_1, x_2, x_3, b_{23}, b_{13}, b_{12}) + \epsilon(-1, -1, 3, -2, -2, 3)$ starting with $\epsilon = 0$, and increase ϵ until $x_2 = x_3$. Call this common value x_{23}, so that $b_{12}, b_{13} \geq 2x_1$, $b_{23} \geq 2x_{23}$ and $4x_{23} \geq x_1 \geq x_{23}$. Then we have

$$x_1 + 2x_{23} + b_{23} + b_{13} + b_{12} + a_1 + a_2 + a_3 = 1$$

and

$$q_1(b_{23} + 2x_{23}) + q_1^2(a_2 + a_3) > \frac{8}{27}$$

$$q_2(b_{13} + x_1 + x_{23}) + q_2^2(a_1 + a_3) > \frac{8}{27}$$

$$q_3(b_{12} + x_1 + x_{23}) + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

We now replace $(x_1, b_{13}) \mapsto (x_1, b_{13}) + \epsilon(1, -1)$ starting with $\epsilon = 0$, and increase ϵ until either $b_{13} = 2x_1$ or $b_{12} = 2x_1$. Without loss of generality, assume that $b_{13} = 2x_1$. Finally, replace $(x_{23}, b_{23}) \mapsto (x_{23}, b_{23}) + \epsilon(1, -2)$ starting with $\epsilon = 0$, and increase ϵ until either $x_{23} = x_1$ or $b_{23} = 2x_{23}$.

In the former case, let x be the common value of x_{23} and x_1. Set $A = 4x$, $B = b_{23} + 2x$, $C = b_{12} + 2x$. Then (assuming $C \geq B$ without loss of generality) we have

System1e $C \geq B \geq A \geq 0$ and

$$\frac{1}{4}A + B + C + a_1 + a_2 + a_3 = 1$$

and

$$q_1 B + q_1^2(a_2 + a_3) > \frac{8}{27}$$

$$q_2 A + q_2^2(a_1 + a_3) > \frac{8}{27}$$

$$q_3 C + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

In the latter case, set $A = 4x_{23}$, $B = 3x_1 + x_{23}$, and $C = b_{12} + x_1 + x_{23}$. Then we have

System1f $C \geq B \geq A \geq 0$ and $B \leq \frac{13}{4}A$ and

$$\frac{7}{12}A + \frac{2}{3}B + C + a_1 + a_2 + a_3 = 1$$

and

$$q_1 A + q_1^2(a_2 + a_3) > \frac{8}{27}$$

$$q_2 B + q_2^2(a_1 + a_3) > \frac{8}{27}$$

$$q_3 C + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

For each of these systems, we carry out the procedure described at the end of Section 2. This process terminates when we have 3 variables apart from q_1, q_2, q_3. Note in **System1f** that if we have $B = \frac{13}{4}A$ then $x_1 = 4x_{23}$

so this is handled by Section 4 and we can avoid analyzing all of these cases. Hence we are left with 20 cases for each system. The case $A = B = C = 0$ is handled as in Section 4.1.

Every case when $A = 0$ other than $A = B = C = 0$ we check by computer as before. Note that **System1e** is identical to **System1f** when $A = B$ as $1 + \frac{1}{4} = \frac{2}{3} + \frac{7}{12}$, so for these cases we only need to check one of the two systems. We report the results of these computations in the Appendix.

5.1. **Remaining cases of Systems1e,f.**

The only cases remaining are the 20 cases, 10 from **System1e** and 10 **System1f**, when three of $A = B, B = C, a_1 = 0, a_2 = 0, a_3 = 0$ occur. Note that $\frac{7}{12}A + \frac{2}{3}B + C + a_1 + a_2 + a_3 \leq \frac{1}{4}A + B + C + a_1 + a_2 + a_3$ when $A \leq B \leq C$, so it suffices to check the 10 cases as above in the following system.

System1f' $C \geq B \geq A \geq 0$ and

$$\frac{7}{12}A + \frac{2}{3}B + C + a_1 + a_2 + a_3 = 1$$

and

$$q_1 A + q_1^2(a_2 + a_3) > \frac{8}{27}$$

$$q_2 B + q_2^2(a_1 + a_3) > \frac{8}{27}$$

$$q_3 C + q_3^2(a_1 + a_2) > \frac{8}{27}.$$

It is vitally important to note that there are solutions to **System1f'**, but we claim there are no solutions in the 10 cases mentioned above.

Suppose there is a solution in one of the cases above, we will derive a contradiction. We have

$$2 = q_1 + q_2 + q_3$$
$$> \frac{16}{27} \frac{1}{A + \sqrt{A^2 + \frac{32}{27}(a_2 + a_3)}} + \frac{1}{B + \sqrt{B^2 + \frac{32}{27}(a_1 + a_3)}} + \frac{1}{C + \sqrt{C^2 + \frac{32}{27}(a_1 + a_2)}}$$
$$\geq \frac{16}{3} \frac{1}{A + B + C + \sqrt{A^2 + \frac{32}{27}(a_2 + a_3)} + \sqrt{B^2 + \frac{32}{27}(a_1 + a_3)} + \sqrt{C^2 + \frac{32}{27}(a_1 + a_2)}}.$$

where the second inequality is by AM-HM, and therefore

$$A + B + C + \sqrt{A^2 + \frac{32}{27}(a_2 + a_3)} + \sqrt{B^2 + \frac{32}{27}(a_1 + a_3)} + \sqrt{C^2 + \frac{32}{27}(a_1 + a_2)} > \frac{8}{3}.$$

We will in fact show that

$$A + B + C + \sqrt{A^2 + \frac{32}{27}(a_2 + a_3)} + \sqrt{B^2 + \frac{32}{27}(a_1 + a_3)} + \sqrt{C^2 + \frac{32}{27}(a_1 + a_2)} \leq \frac{8}{3}$$

in each of the 10 cases, yielding a contradiction.

Suppose first we are in one of the three cases where $A = B = C$. Then $\frac{9}{4}A + a_1 + a_2 + a_3 = 1$, and by Jensen's inequality applied to $\sqrt{x}$, it suffices to show that

$$3A + \sqrt{3}\sqrt{3A^2 + \frac{64}{27}(a_1 + a_2 + a_3)} \leq \frac{8}{3}.$$

As $A \leq \frac{4}{9}$, we have $\frac{8}{3} \geq 3A$. As $a_1 + a_2 + a_3 = 1 - \frac{9}{4}A$, this inequality is

$$3A + |\frac{8}{3} - 3A| \leq \frac{8}{3},$$

and as $\frac{8}{3} \geq 3A$, both sides are equal.

If $a_1 = a_2 = a_3 = 0$, then we have $2(A + B + C) \leq \frac{8}{3}(\frac{7}{12}A + \frac{2}{3}B + C) = \frac{8}{3}$ by the Chebyshev sum inequality.

In the remaining cases, we have one of $A = B$ and $B = C$. We have dealt with all cases where the maximum occurs on the boundary except when $A = 0$, so we can assume the maximum is attained in the interior of the domain or $A = 0$.

When $A = B$, we have by symmetry two distinct cases, when $a_1 = a_2 = 0$ and when $a_1 = a_3 = 0$.

When $a_1 = a_2 = 0$ we have $0 \leq A \leq C$, $\frac{5}{4}A + C + a_3 = 1$ and we want to show

$$2A + 2C + 2\sqrt{A^2 + \frac{32}{27}a_3} \leq \frac{8}{3}.$$

By Lagrange multipliers, if the triple (A, C, a_3) attains a maximum on the interior of the domain then the gradient of the constraint $(\frac{5}{4}, 1, 1)$ is parallel to the gradient $(2 + \frac{2A}{\sqrt{A^2 + \frac{32}{27}a_3}}, 2, \frac{32}{27}\frac{1}{\sqrt{A^2 + \frac{32}{27}a_3}})$ of the objective function, so we have

$$\frac{4}{5}(2 + \frac{2A}{\sqrt{A^2 + \frac{32}{27}a_3}}) = 2$$

$$\frac{32}{27}\frac{1}{\sqrt{A^2 + \frac{32}{27}a_3}} = 2.$$

But then $\sqrt{A^2 + \frac{32}{27}a_3} = \frac{16}{27}$, so $A = \frac{4}{27}$, $a_3 = \frac{5}{18}$, and so $C = \frac{29}{54}$. These values make the left hand side $\frac{23}{9} < \frac{8}{3}$. If the maximum occurs when $A = 0$, then the inequality is equivalent to $(\frac{4}{3} - C)^2 \geq \frac{32}{27}(1 - C)$, which is true.

When $a_1 = a_3 = 0$ we have $0 \leq A \leq C$, $\frac{5}{4}A + C + a_2 = 1$, and we want to show

$$3A + C + \sqrt{A^2 + \frac{32}{27}a_2} + \sqrt{C^2 + \frac{32}{27}a_2} \leq \frac{8}{3}.$$

If the maximum is attained in the interior of the domain, by Lagrange multipliers

$$3 + \frac{A}{\sqrt{A^2 + \frac{32}{27}a_2}} = \frac{5}{4} + \frac{5}{4}\frac{C}{\sqrt{C^2 + \frac{32}{27}a_2}},$$

which cannot happen as $\frac{C}{\sqrt{C^2 + \frac{32}{37}a_2}} \leq 1$. If the maximum occurs when $A = 0$, then the left hand side is at most $C + \sqrt{2C^2 + \frac{128}{27}a_2}$, and the inequality then becomes $(\frac{8}{3} - C)^2 \geq 2C^2 + \frac{128}{27}(1 - C)$, which is true as $C \leq 1$.

Now, when $B = C$, we have by symmetry two distinct cases, when $a_2 = a_3 = 0$ and when $a_1 = a_3 = 0$.

When $a_2 = a_3 = 0$, we have $0 \leq A \leq C$, $\frac{7}{12}A + \frac{5}{3}C + a_1 = 1$, and we want to show

$$2A + 2C + 2\sqrt{C^2 + \frac{32}{27}a_1} \leq \frac{8}{3}.$$

If the maximum is attained in the interior of the domain, by Lagrange multipliers

$$-\frac{120}{21} + 2 + \frac{2C}{\sqrt{C^2 + \frac{32}{27}a_1}} = 0$$

which cannot happen as $\frac{2C}{\sqrt{C^2 + \frac{32}{27}a_1}} \leq 2$. If the maximum occurs when $A = 0$, then the inequality is equivalent to $(\frac{4}{3} - C)^2 \geq C^2 + \frac{32}{27}(1 - \frac{5}{3}C)$, which is true as $C \leq \frac{3}{5}$.

Finally, when $a_1 = a_3 = 0$, we have $0 \leq A \leq C$, $\frac{7}{12}A + \frac{5}{3}C + a_2 = 1$, and we want to show

$$A + 3C + \sqrt{A^2 + \frac{32}{27}a_2} + \sqrt{C^2 + \frac{32}{27}a_2} \leq \frac{8}{3}.$$

By Jensen's inequality applied to $\sqrt{x}$, it suffices to show that

$$A + 3C + \sqrt{2A^2 + 2C^2 + \frac{128}{27}a_2} \leq \frac{8}{3}.$$

As $A + 3C \leq \frac{8}{3}(\frac{7}{12}A + \frac{5}{3}C) \leq \frac{8}{3}$, we can move $A + 3C$ to the right hand side, square the equation, and then replace a_2 with $1 - \frac{7}{12}A - \frac{5}{3}C$ to get

$$(A + 3C - \frac{8}{3})^2 - (2A^2 + 2C^2 + \frac{128}{27}(1 - \frac{7}{12}A - \frac{5}{3}C)) \geq 0.$$

If the minimum is attained in the interior of the domain, then we have the partial derivatives with respect to A and C vanishing. This yields

$$2(A + 3C - \frac{8}{3}) - 4A + \frac{224}{81} = 0$$
$$6(A + 3C - \frac{8}{3}) - 4C + \frac{640}{81} = 0,$$

so $A = \frac{16}{81}, C = \frac{40}{81}$. Evaluating at this point yields $\frac{256}{2187} \geq 0$. The boundary case $A = C$ is identical to the case $A = B = C$ when $a_1 = a_3 = 0$. The boundary case $A = 0$ is equivalent to $(\frac{8}{3} - 3C)^2 \geq 2C^2 + \frac{128}{27}(1 - \frac{5}{3}C)$ for $0 \leq C \leq \frac{3}{5}$, and the boundary case $a_2 = 0$ is equivalent to $(\frac{8}{3} - 3C - (\frac{12}{7} - \frac{20}{7}C))^2 \geq 2C^2 + 2(\frac{12}{7} - \frac{20}{7}C)^2$ for $\frac{4}{9} \leq C \leq \frac{3}{5}$ (the lower bound on C when $a_2 = 0$ follows from $1 = \frac{7}{12}A + \frac{5}{3}C \leq (\frac{7}{12} + \frac{5}{3})C = \frac{9}{4}C$).

6. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 using Lemma 2.1.

We can assume every vertex has positive degree. Order the vertices $v_1, \ldots, v_n$ such that $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$. Let V^{high} consist of the $t = m^\alpha$ vertices of highest degree with some fixed $0 < \alpha < \frac{1}{3}$. Then

$$m^\alpha d(v_{t+1}) < \sum_{v \in \mathcal{H}} d(v) = 3m,$$

so

$$d(v_{t+1}) \leq 3m^{1-\alpha}.$$

Hence in particular,

$$\sum_{i=t+1}^{n} d(v_i)^2 < 3m^{1-\alpha} \sum_{i=t+1}^{n} d(v_i) \leq 9m^{2-\alpha}.$$

Recall $x_1, x_2, x_3, b_{12}, b_{13}, b_{23}$ were defined in the introduction. Further define a_i for $i \in \{1, 2, 3\}$ to be the number of edges in $E(\mathcal{H})^1$ such that the high degree vertex lies in V_i^{high}, and let c be the number of edges in $E(\mathcal{H})^0$.

Apply Lemma 2.1 to all of our variables scaled by $\frac{1}{m - E(V^{\text{high}})}$ to get probabilities $p_i = 1 - q_i$. Place the low degree vertices independently into V_i with probability p_i. Then with $d(V_i)$ the number of edges with at least one vertex in V_i, we have $\mathbb{E}(d(V_i)) \geq \frac{19}{27}(m - E(V^{\text{high}}))$.

We recall that the Azuma-Hoeffding inequality [8] asserts that if $X_1, \ldots, X_n$ are independent random variables with values in $\{1, \ldots, k\}$, and $f : \{1, \ldots, k\}^n \to \mathbb{N}$ such that $|f(Y) - f(Y')| \leq d_i$ if Y and Y' differ only on the i'th coordinate, then for $t \geq 0$ we have $\mathbb{P}(f(X_1, \ldots, X_n) \leq \mathbb{E}(f(X_1, \ldots, X_n)) - t) \leq e^{-\frac{2t^2}{\sum d_i^2}}$.

By the Azuma-Hoeffding inequality with $k = 3$, we have

$$\mathbb{P}(d(V_i) < \mathbb{E}(d(V_i)) - z) \leq e^{\frac{-2z^2}{\sum_{i=t+1}^{n} d(v_i)^2}} < e^{-\frac{2z^2}{9m^{2-\alpha}}}.$$

Taking $z = \sqrt{(9/2)\ln 3}\, m^{1-\frac{\alpha}{2}}$ yields $\mathbb{P}(d(V_i) < \mathbb{E}(d(V_i)) - z) < \frac{1}{3}$. Hence there is a partition which has $d(V_i) > \frac{19}{27}(m - E(V^{\text{high}})) - z$ simultaneously for $i = 1, 2, 3$. Because $E(V^{\text{high}}) = O(m^{3\alpha})$ and $z = O(m^{1-\frac{\alpha}{2}})$, taking $\alpha = \frac{2}{7}$ yields $d(V_i) > \frac{19}{27}m - O(m^{6/7})$ simultaneously for $i = 1, 2, 3$ as desired.

Acknowledgements. We would like to thank our advisor Béla Bollobás for all of his support, and the anonymous referees for helpful comments.

Appendix A. Results of Computation

Tables 1–6 show the results of computation. We denote by ϵ the side length of the cubes used in the interval method (when applicable). We use $*$ to indicate that the minimum value was proved in the body of the paper. We record a proven lower bound in the "Lower Bound" column. Finally, we note that when $B = A$ in **System1f**, the case is identical to the corresponding case in **System1e**.

Table 1. System1a

$x_{23} = 0$	$b_{13} = 8x_{23}$	$b_{23} = 2x_{23}$	$a_1 = 0$	$a_2 = 0$	$a_3 = 0$	ϵ	Lower Bound
✓	✓	✓					2*
✓	✓		✓			0.002	2.078
✓	✓			✓		0.002	2.077
✓	✓				✓	0.002	2.077
✓		✓	✓			0.002	2.077
✓		✓		✓		0.002	2.078
✓		✓			✓	0.002	2.077
✓			✓	✓		0.002	2.085
✓			✓		✓	0.002	2.086
✓				✓	✓	0.002	2.086
	✓	✓	✓			0.001	2.005
	✓	✓		✓		0.002	2.033
	✓	✓			✓	0.002	2.033
	✓		✓	✓		0.002	2.057
	✓		✓		✓	0.002	2.057
	✓			✓	✓	0.002	2.043
		✓	✓	✓		0.002	2.045
		✓	✓		✓	0.002	2.044
		✓		✓	✓	0.002	2.041
			✓	✓	✓	0.002	2* or 2.046

TABLE 2. System1b

$x_{23}=0$	$b_{13}=2x_1$	$x_1=4x_{23}$	$a_1=0$	$a_2=0$	$a_3=0$	ϵ	Lower Bound
✓	✓	✓					2∗
✓	✓		✓			0.002	2.042
✓	✓			✓		0.002	2.069
✓	✓				✓	0.002	2.069
✓		✓	✓			0.002	2.077
✓		✓		✓		0.002	2.078
✓		✓			✓	0.002	2.077
✓			✓	✓		0.002	2.072
✓			✓		✓	0.002	2.072
✓				✓	✓	0.002	2.070
	✓	✓	✓			0.001	2.005
	✓	✓		✓		0.002	2.033
	✓	✓			✓	0.002	2.033
	✓		✓	✓		0.002	2.026
	✓		✓		✓	0.002	2.026
	✓			✓	✓	0.002	2.024
		✓	✓	✓		0.002	2.045
		✓	✓		✓	0.002	2.044
		✓		✓	✓	0.002	2.041
			✓	✓	✓	0.002	2∗ or 2.025

TABLE 3. System1c

$x_3=0$	$x_2=x_3$	$4x_2=x_1$	$a_1=0$	$a_2=0$	$a_3=0$	ϵ	Lower Bound
✓	✓	✓					2∗
✓	✓		✓			0.002	2.042
✓	✓			✓		0.002	2.069
✓	✓				✓	0.002	2.069
✓		✓	✓			0.001	2.019
✓		✓		✓		0.002	2.036
✓		✓			✓	0.002	2.037
✓			✓	✓		0.002	2.027
✓			✓		✓	0.002	2.028
✓				✓	✓	0.002	2.026
	✓	✓	✓			0.001	2.005
	✓	✓		✓		0.002	2.033
	✓	✓			✓	0.002	2.033
	✓		✓	✓		0.002	2.026
	✓		✓		✓	0.002	2.026
	✓			✓	✓	0.002	2.024
		✓	✓	✓		0.001	2.042
		✓	✓		✓	0.001	2.042
		✓		✓	✓	0.001	2.042
			✓	✓	✓	0.001	2∗ or 2.033

Table 4. System1d

$b_{12}=8x_2$	$x_2=x_3$	$x_3=0$	$a_1=0$	$a_2=0$	$a_3=0$	ϵ	Lower Bound
✓	✓	✓					2∗
✓	✓		✓			0.002	2.077
✓	✓			✓		0.002	2.077
✓	✓				✓	0.002	2.078
✓		✓	✓			0.001	2.019
✓		✓		✓		0.002	2.036
✓		✓			✓	0.002	2.037
✓			✓	✓		0.002	2.047
✓			✓		✓	0.002	2.047
✓				✓	✓	0.002	2.041
	✓	✓	✓			0.001	2.005
	✓	✓		✓		0.002	2.033
	✓	✓			✓	0.002	2.033
	✓		✓	✓		0.002	2.044
	✓		✓		✓	0.002	2.045
	✓			✓	✓	0.002	2.041
		✓	✓	✓		0.001	2.042
		✓	✓		✓	0.001	2.042
		✓		✓	✓	0.001	2.042
			✓	✓	✓	0.001	2∗ or 2.042

Table 5. System1e

$A=0$	$B=A$	$C=B$	$a_1=0$	$a_2=0$	$a_3=0$	ϵ	Lower Bound
✓	✓	✓					2∗
✓	✓		✓			0.002	2.077
✓	✓			✓		0.002	2.077
✓	✓				✓	0.002	2.078
✓		✓	✓			0.002	2.075
✓		✓		✓		0.002	2.076
✓		✓			✓	0.002	2.076
✓			✓	✓		0.002	2.086
✓			✓		✓	0.002	2.085
✓				✓	✓	0.002	2.084
	✓	✓	✓				2∗
	✓	✓		✓			2∗
	✓	✓			✓		2∗
	✓		✓	✓			2∗
	✓		✓		✓		2∗
	✓			✓	✓		2∗
		✓	✓	✓			2∗
		✓	✓		✓		2∗
		✓		✓	✓		2∗
			✓	✓	✓		2∗

TABLE 6. System1f

$A = 0$	$B = A$	$C = B$	$a_1 = 0$	$a_2 = 0$	$a_3 = 0$	ϵ	Lower Bound
✓	✓	✓					2*
✓	✓		✓			0.002	2.077
✓	✓			✓		0.002	2.077
✓	✓				✓	0.002	2.078
✓		✓	✓			0.002	2.075
✓		✓		✓		0.002	2.076
✓		✓			✓	0.002	2.076
✓			✓	✓		0.002	2.086
✓			✓		✓	0.002	2.085
✓				✓	✓	0.002	2.084
	✓	✓	✓				2*
	✓	✓		✓			2*
	✓	✓			✓		2*
	✓		✓	✓			2*
	✓		✓		✓		2*
	✓			✓	✓		2*
		✓	✓	✓			2*
		✓	✓		✓		2*
		✓		✓	✓		2*
			✓	✓	✓		2*

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