MULTICOLOR CHAIN AVOIDANCE IN THE BOOLEAN LATTICE

HUNTER SPINK AND MARIUS TIBA

ABSTRACT. Given a collection of colored chain posets, we estimate the number of colored subsets of the boolean lattice which avoid all chains in the collection. We do this by applying the hypergraph container lemma recursively, with different uniformities at each stage, using a balanced supersaturation result for a certain non-uniform hypergraph encoding forbidden configurations.

1. INTRODUCTION

The method of hypergraph containers, recently introduced by Balogh, Morris and Samotij [BMS15] and independently by Saxton and Thomason [ST15], is an essential tool for counting independent sets in hypergraphs. Many natural problems can be phrased in this way, with the most direct applications toward determining the number of graphs on n vertices which avoid a collection of subgraphs. We refer the reader to the survey [BMS18].

Recently, Balogh, Treglown and Wagner [BTW16] showed that graph containers could be used in the boolean lattice $(\mathcal{P}([n]), \subseteq)$ to give an alternate proof of Kleitman's result [Kle69] counting the number of antichains in $\mathcal{P}([n])$ (we note that the proof in [Kle69] precedes the notion of graph containers, but has similar ideas). Using the hypergraph container algorithm, independently Collares and Morris [CM16] and Balogh, Mycroft, and Treglown [BMT14] were able to further count the number of antichains in a random subset of $\mathcal{P}([n])$, from which they were able to deduce the approximate size of the largest antichain therein.

Because every k-chain free set can be partitioned into $k-1$ antichains, the graph container lemma (i.e. for 2-uniform hypergraphs) suffices not only to count k-chain free sets, but also to create a small collection of small sized containers for k-chain free sets via a product construction. However, [CM16] directly constructs a set of hypergraph containers without exploiting this observation as an application of the recent advances in the hypergraph container lemma through balanced supersaturation results.

In this paper we answer analogous questions in a weighted colored setting by building on [CM16]'s demonstration of the hypergraph container lemma through balanced supersaturation in $\mathcal{P}([n])$. Fix colors $1, 2, \ldots, m$. Given a poset $\mathcal{P}$, a *coloring* of $\mathcal{P}$ is a function $c : \mathcal{P} \to \{1, \ldots, m\}$. A *colored poset* $(\mathcal{P}, c)$ is a poset equipped with a coloring. A *colored subset* of a poset $\mathcal{P}$ (resp. colored poset $(\mathcal{P}, c')$) is a pair (S, c) with $S \subset \mathcal{P}$ and c a coloring of S (resp. the coloring of S agreeing with c'). Let $\mathcal{G}$ be a collection of colored chain posets of lengths at least 2. Denoting disjoint union by $\sqcup$, we write

$$\mathcal{G} := \mathcal{G}_2 \sqcup \mathcal{G}_3 \sqcup \ldots \sqcup \mathcal{G}_k$$

where $\mathcal{G}_i$ contains exclusively chains of length i. We say that a colored subset (S, c) of $\mathcal{P}([n])$ *avoids all configurations from* $\mathcal{G}$, or is *valid with respect to* $\mathcal{G}$, if no element of $\mathcal{G}$ appears as a colored subchain of (S, c). The present work is motivated by the following questions.

Question 1.1. What is the cardinality of the collection $\Lambda(\mathcal{G}, n)$ (where the dependency on the number of colors m is suppressed here and throughout the paper) of validly colored subsets (S, c) of $\mathcal{P}([n])$ with respect to $\mathcal{G}$?

Question 1.2. Let $p_1, \ldots, p_m \in (0, 1]$ be such that $\sum_{i=1}^m p_i \leq 1$. If we color each element of $\mathcal{P}([n])$ independently with color c_i with probability p_i and leave it uncolored with probability $1 - \sum p_i$, then what is the expected number of validly colored subsets (S, c) of (the colored part of) $\mathcal{P}([n])$ with respect to $\mathcal{G}$?

Multicolored hypergraph container problems were only considered quite recently in the work of Falgas-Ravry, O'Connell, and Uzzell [FOU19]. There it was shown that for a wide variety of colored configuration avoidance problems, if there is a validly colored subset using all but a $o(1)$ proportion of the vertices, then the number of validly colored subsets can be estimated quite precisely (see Theorem 1.8).

The questions we consider are the first instances of colored hypergraph container problems that we are aware of which work in the presence of a "sparse" extremal example. A separate interesting feature is that the hypergraph we work with is not uniform and no uniformity dominates, so when we iteratively apply the container algorithm we may have to use potentially different uniformities at each stage.

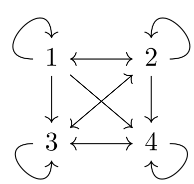
Example 1.3. We now describe some instructive examples. A colored chain with colors $a_1, \ldots, a_\ell$ in increasing order will be denoted $a_1 \prec \ldots \prec a_\ell$.

(1) Suppose we have 1 color, i.e. $m = 1$, and that we have only one forbidden chain $\mathcal{G} = \{(\underbrace{1 \prec 1 \prec \ldots \prec 1}_{k})\}$. Then $\Lambda(\mathcal{G}, n)$ is the collection of k-chain free sets of $\mathcal{P}([n])$, and by [Kle69, BTW16, CM16] we have

$$|\Lambda(\mathcal{G}, n)| = 2^{(k-1)\binom{n}{n/2}(1+o(1))}.$$

(2) Suppose that we have 4 colors, and let $\mathcal{G}$ be the 13 colored 2-chains

$\mathcal{G} = \mathcal{G}_2 =$



, where $i \to j$ stands for $(i \prec j)$.

Equivalently for $(S, c) \in \Lambda(\mathcal{G}, n)$, if $x, y \in S$ with $x \subsetneq y$, then we have $(c(x), c(y)) \in \{(3, 1), (4, 1), (4, 2)\}$.

Notably in this example $\Lambda(\mathcal{G}, n)$ contains two fundamentally different large families of valid configurations.

- Consider the family of colored subsets (S, c) such that the elements of S have sizes in $\{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor - 1\}$, and

$$c(x) = \begin{cases} 1 & |x| = \lfloor n/2 \rfloor \\ 3 \text{ or } 4 & |x| = \lfloor n/2 \rfloor - 1. \end{cases}$$

In other words, we can color all sets of size $\lfloor n/2 \rfloor$ with either 1 or nothing, and all sets of size $\lfloor n/2 \rfloor - 1$ with $3, 4$ or nothing.
- Alternatively, we can color all sets of size $\lfloor n/2 \rfloor$ with either $1, 2$ or nothing, and all sets of size $\lfloor n/2 \rfloor - 1$ with either 4 or nothing.

In either case, the size of the family is

$$2^{\binom{n}{n/2}(1+o(1))} \cdot 3^{\binom{n}{n/2}(1+o(1))} = 6^{\binom{n}{n/2}(1+o(1))}.$$

As it turns out, by Theorem 1.10 it follows that this is also an upper bound to $|\Lambda(\mathcal{G}, n)|$.

(3) Suppose we have 2 colors, and $\mathcal{G} = \mathcal{G}_2 = \{(1 \prec 2)\}$. Then

$$|\Lambda(\mathcal{G}, n)| \geq 2^{2^n(1+o(1))},$$

obtained by coloring each vertex either with 1 or nothing. As we will shortly see, this is also an upper bound by [FOU19].

In general, the "dense" cases of Question 1.1 and Question 1.2 (see Remark 1.9) are solved by [FOU19]. This occurs exactly when $\mathcal{G}$ does not contain a monochromatic chain of every color. We are able to complete the analysis of these questions in the remaining "sparse" cases.

1.1. Preliminary definitions and main results.

We recall from [FOU19] some basic definitions of multicolor hypergraphs in our context.

Definition 1.4. For a poset $\mathcal{P}$, a *template* is a function

$$T : \mathcal{P} \to \mathcal{P}(\{1, \ldots, m\})$$

from $\mathcal{P}$ to subsets of colors. Say that a template T is *supported on* $A \subset \mathcal{P}$ if $T(x) = \emptyset$ whenever $x \notin A$, and define $\mathrm{Supp}(T)$ to be the smallest set on which T is supported.

Definition 1.5. For a template T on a poset $\mathcal{P}$, we say a colored subset (A, c) of $\mathcal{P}([n])$ is *contained in* T if $c(x) \in T(x)$ for all $x \in A$ (note that such an A is contained in $\mathrm{Supp}(T)$). We say that T is *valid with respect to* $\mathcal{G}$ if every colored subset contained in T is valid with respect to $\mathcal{G}$.

Definition 1.6. For a template T on a poset $\mathcal{P}$, denote by

$$\omega(T) = \sum_{x \in \mathcal{P}} \log(1 + |T(x)|),$$

so that $e^{\omega(T)}$ is the number of colored subsets contained in T.

We will typically consider tempates on the poset $\mathcal{P} = \mathcal{P}([n])$.

Remark 1.7. The reason we consider valid templates is that they provide a lower bound on the number of valid configurations

$$|\Lambda(\mathcal{G}, n)| \geq \max_{T \text{ valid}} \prod_{x \in \mathcal{P}([n])} (1 + |T(x)|) = e^{\max_{T \text{ valid}} \omega(T)}.$$

One would hope that this is the correct bound up to a $(1 + o(1))$ factor in the exponent. The following theorem of [FOU19], gives the sharp exponent in the "dense" case.

Theorem 1.8. [FOU19, Corollary 3.34] Define the *maximal entropy* of $\mathcal{G}$ to be

$$\pi(\mathcal{G}) = \limsup_{n\to\infty} \frac{1}{2^n} \max_{T \text{ valid}} \omega(T).$$

We have

$$|\Lambda(\mathcal{G}, n)| = e^{2^n(\pi(\mathcal{G})+o(1))}.$$

Remark 1.9. As mentioned earlier, this theorem correctly estimates $|\Lambda(\mathcal{G}, n)|$ up to a $1 + o(1)$ factor in the exponent when $\pi(\mathcal{G}) > 0$. We refer to this as the *dense* case, and it occurs precisely when $\mathcal{G}$ contains a monochromatic chain of every color. In the *sparse* case when $\pi(\mathcal{G}) = 0$, the upper bound given by Theorem 1.8 is trivial and does not estimate the correct exponent up to a $1 + o(1)$ factor.

We solve Question 1.1 in the sparse cases by estimating $|\Lambda(\mathcal{G}, n)|$ up to a $1+o(1)$ factor in the exponent. This occurs when $\pi(\mathcal{G}) = 0$, i.e. $\mathcal{G}$ contains a monochromatic chain of every color. Note that in this case, there exists $L = L(\mathcal{G})$ such that no validly colored set (S, c) contains a chain of length L, and we may easily deduce a crude upper bound of $|\Lambda(\mathcal{G}, n)| \leq (m + 1)^{(L-1)\binom{n}{n/2}(1+o(1))}$ by using the L-chain containers constructed from either of [BTW16, CM16].

The following is one of our main theorems, solving the sparse cases of Question 1.1.

Theorem 1.10. Suppose that $\mathcal{G}$ contains a monochromatic chain of every color, and define the constant (independent of n)

$$\omega_{crit} = \max_T \{\omega(T) \mid T \text{ a template on a chain poset, valid with respect to } \mathcal{G}\}.$$

Then we have

$$|\Lambda(\mathcal{G}, n)| = e^{\omega_{crit}\binom{n}{n/2}(1+o(1))}.$$

Remark 1.11. Note that to compute the constant ω_{crit}, we only need to evaluate $\omega(T)$ for T ranging over the finite collection of valid templates on chain posets of length at most $L = L(\mathcal{G})$ as above, assigning each element a non-empty set.

As we will see later in Proposition 2.5, we can determine the exact maximum of $\omega(T)$ for valid templates T on $\mathcal{P}([n])$, and this value is $\omega_{crit}\binom{n}{n/2}(1 + o(1))$. The lower bound in Theorem 1.10 will follow as in Remark 1.7.

Example 1.12.

- Suppose we are in the situation of Example 1.3 (1). The maximum ω_{crit} is attained by the template T on a length $k - 1$ chain poset, sending every element to the set $\{1\}$. Then $\omega_{crit} = \omega(T) = \log 2^{k-1}$, and $|\Lambda(\mathcal{G}, n)| = 2^{(k-1)\binom{n}{n/2}(1+o(1))}$, recovering the results of [Kle69, BTW16, CM16].
- Suppose we are in the situation of Example 1.3 (2). Then the following two templates T on chain posets realize ω_{crit}. For $x \prec y$ a chain poset of length 2, the first such template has $T(x) = \{3, 4\}$, $T(y) = \{1\}$, and the second has $T(x) = \{4\}$ and $T(y) = \{1, 2\}$, attaining $\omega(T) = \omega_{crit} = \log 6$. We thus have $|\Lambda(\mathcal{G}, n)| = 6^{\binom{n}{n/2}(1+o(1))}$.

The answer to Question 1.2 requires a weighted version of $\omega(T)$.

Definition 1.13. Given $\mathcal{G}$ that contains a monochromatic chain of every color, $\overline{\beta} = (\beta_1, \ldots, \beta_m) \in (\mathbb{R}_{>0})^m$, and T a template on a poset $\mathcal{P}$, we denote by

$$|T(x)|_{\overline{\beta}} = \sum_{i \in T(x)} \beta_i$$

$$\omega(\overline{\beta}, T) = \sum_{x \in \mathcal{P}} \log(1 + |T(x)|_{\overline{\beta}})$$

$$\omega_{crit}(\overline{\beta}) = \max_T \{\omega(\overline{\beta}, T) \mid T \text{ a template on a chain poset, valid with respect to } \mathcal{G}\}.$$

The constant $\omega_{crit}(\overline{\beta})$ is also very easy to compute analogously to ω_{crit}, and yields the critical exponent in the following theorem.

Theorem 1.14. Suppose that $\mathcal{G}$ contains a monochromatic chain of every color. With the probabilistic setup of Question 1.2, denoting $\overline{p} = (p_1, \ldots, p_m)$ and V the number of validly colored subsets (S, c) of (the colored part of) $\mathcal{P}([n])$ with respect to $\mathcal{G}$, we have

$$\mathbb{E}(V) = e^{\omega_{crit}(\overline{p})\binom{n}{n/2}(1+o(1))}.$$

As before, the methods from [FOU19] analogously answer the dense cases where there is some color without a monochromatic forbidden chain. Theorem 1.14 thus completes the analysis of Question 1.2 in the sparse case.

For the remainder of the paper, we write colored sets as S rather than (S, c), suppressing the coloring c.

Acknowledgements: We would like to thank our advisor Béla Bollobás for his many years of support, as well as the anonymous referees for helping clarify the exposition of the paper.

2. Strategy and auxiliary results

In this section we outline the strategy of the proof and present some auxiliary results. We start by formulating a weighted version of Theorem 1.10 and Theorem 1.14 which encompasses both of them. Throughout the rest of the paper we shall always assume that $\mathcal{G}$ contains a monochromatic chain of each color and refer to such a $\mathcal{G}$ as *sparse*. In this section, we let $\mathcal{G} \coloneqq \mathcal{G}_2 \sqcup \ldots \sqcup \mathcal{G}_k$ by a fixed collection of forbidden colored chains with colors $1, \ldots, m$, and we let

$$\overline{\beta} = (\beta_1, \ldots, \beta_m) \in (\mathbb{R}_{>0})^m$$

be a fixed sequence of positive real weights.

Definition 2.1. We define a measure μ on colored subsets of $\mathcal{P}([n])$ by assigning to a subset $S \subset \mathcal{P}([n])$ with a coloring $c\colon S \to \{1, \ldots, m\}$, the weight

$$\mu(\overline{\beta}, S) = \prod_{x \in S} \beta_{c(x)}$$

and extending it additively. In particular, for a collection Λ of colored subsets of $\mathcal{P}([n])$, we have

$$\mu(\overline{\beta}, \Lambda) = \sum_{S \in \Lambda} \mu(\overline{\beta}, S).$$

Note that when $\beta_i = 1$ for all i we have that $\mu(\overline{\beta}, S) = 1$ for every colored subset S, and $\mu(\overline{\beta}, \Lambda) = |\Lambda|$. We may now state the weighted reformulation of our main theorems.

Theorem 2.2. Suppose that $\mathcal{G}$ is sparse, then we have

$$\mu(\overline{\beta}, \Lambda(\mathcal{G}, n)) = e^{\omega_{crit}(\overline{\beta})\binom{n}{n/2}(1+o(1))}.$$

Remark 2.3. Theorem 2.2 specializes to Theorem 1.10 when all $\beta_i = 1$, and specializes to Theorem 1.14 when $\beta_i = p_i$.

2.1. **Proof of the lower bound of Theorem 2.2.** We now prove two propositions which we shall use to prove the lower bound in Theorem 2.2. The first proposition relates the weight of a template to the measure of the collection of colored subsets contained in the template. The second proposition relates the maximum weight of a template to the critical weight.

Proposition 2.4. Given a template T, let Λ_T be the collection of colored subsets contained in T. Then

$$\mu(\overline{\beta}, \Lambda_T) = e^{\omega(\overline{\beta}, T)}.$$

Proof. Note that

$$e^{\omega(\overline{\beta}, T)} = \prod_{x \in \mathrm{Supp}(T)} (1 + |T(x)|_{\overline{\beta}}) = \prod_{x \in \mathrm{Supp}(T)} (1 + \sum_{i \in T(x)} \beta_i),$$

and expanding out the product yields $\mu(\overline{\beta}, \Lambda_T)$. □

Proposition 2.5. The maximum value of $\omega(\overline{\beta}, T)$ where T is a valid template is attained for some T with $\mathrm{Supp}(T)$ a consecutive block of layers of $\mathcal{P}([n])$ which intersects the middle layer(s), and with the property that $T(x)$ depends only on the size of x. In particular, we have that

$$\max_{\text{T valid}} \omega(\overline{\beta}, T) = (1 + O(\frac{1}{n}))\omega_{crit}(\overline{\beta})\binom{n}{n/2}.$$

Proof. Our strategy will be to consider a valid template T' which maximizes $\omega(\overline{\beta}, T')$, and construct from it another valid template T that satisfies the conclusions of Proposition 2.5. Choose a uniformly random maximal chain C and consider the random variable

$$Z(C) = \sum_{x \in C} \binom{n}{|x|} \log(1 + |T'(x)|_{\overline{\beta}}).$$

By linearity of expectation, it is easy to see that

$$\mathbb{E}Z(C) = \omega(\overline{\beta}, T').$$

Therefore, there exists a chain C such that $Z(C) \geq \omega(\overline{\beta}, T')$. But then the template T defined by $T(x) = T'(y)$ with $y \in C$ the unique element such that $|x| = |y|$ satisfies

$$\omega(\overline{\beta}, T) = Z(C) \geq \omega(\overline{\beta}, T').$$

By construction T is valid since $\mathcal{G}$ consists exclusively of chains, and by maximality of T', $\omega(\overline{\beta}, T) = \omega(\overline{\beta}, T')$ is maximal. Clearly the maximality of such a T further implies that $\mathrm{Supp}(T)$ is a consecutive block of layers containing the middle layer. □

We are now ready to prove the lower bound in Theorem 2.2.

Proof of the lower bound in Theorem 2.2. Let T be the extremal template from Proposition 2.5 and let Λ_T be the collection of colored subsets contained in T. Then by Proposition 2.4 and Proposition 2.5 we have that

$$\mu(\overline{\beta}, \Lambda(\mathcal{G}, n)) \geq \mu(\overline{\beta}, \Lambda_T) = e^{\omega(\overline{\beta}, T)} = e^{(1+O(\frac{1}{n}))\omega_{crit}(\overline{\beta})\binom{n}{n/2}},$$

which gives the desired lower bound. □

2.2. **Proof of the upper bound of Theorem 2.2 assuming a balanced supersaturation result.** Now we describe the outline of the proof of the upper bound in Theorem 2.2. The main thrust of the proof is identical to that of [CM16]. The key new ideas are to create a balanced supersaturation result that works for templates and to implement the container lemma in a way which handles simultaneously the various uniformities of $\mathcal{G}$.

Our goal will be to find a collection $\mathcal{C}$ of $e^{o(1)\binom{n}{n/2}}$ templates with each template $T \in \mathcal{C}$ having $w(\overline{\beta}, T) \leq (\omega_{crit}(\overline{\beta}) + o(1))\binom{n}{n/2}$ such that every validly colored subset of $\mathcal{P}([n])$ is contained in some template $T \in \mathcal{C}$. Then by a union bound we can conclude Theorem 2.2.

To accomplish this, we will use the following hypergraph container lemma.

Given a hypergraph $\mathcal{H}$, we denote $v(\mathcal{H})$ for the vertices of $\mathcal{H}$, $e(\mathcal{H})$ for the edges of $\mathcal{H}$, and we recall for $A \subset v(\mathcal{H})$ the standard notations $d_{\mathcal{H}}(A)$ for the number of hyperedges of $\mathcal{H}$ which contain A, and jth codegree $\Delta_j(\mathcal{H}) = \max_{|A|=j} d_{\mathcal{H}}(A)$.

Lemma 2.6. [BMS15, ST15] For every $K \in \mathbb{N}$ and $c > 0$ there exists $\epsilon > 0$ such that the following holds. Let $\tau \in (0, 1)$ and suppose that $\mathcal{H}$ is a K-uniform hypergraph on N vertices such that for $1 \leq j \leq K$ we have

$$\Delta_j(\mathcal{H}) \leq c \cdot \tau^{j-1} \frac{e(\mathcal{H})}{N}.$$

Then there exists a family $\mathcal{C}$ of subsets of $v(\mathcal{H})$, and a function $f \colon \mathcal{P}(v(\mathcal{H})) \to \mathcal{C}$ such that

(1) For every $I \in \mathcal{I}(\mathcal{H})$, there exists $F \subset I$ with $|F| \leq K\tau N$ and $I \subset F \cup f(F)$
(2) $|C| \leq (1 - \epsilon)N$ for every $C \in \mathcal{C}$.

In order to use the hypergraph container lemma we translate between validly colored subsets of $\mathcal{P}([n])$ and independent subsets of a certain hypergraph. We consider the following ambient non-uniform hypergraph $\mathcal{A}$ defined by

$$v(\mathcal{A}) = \mathcal{P}([n]) \times \{1, \dots, m\},$$

$$e(\mathcal{A}) = \bigcup_{\ell=2}^{k} \{\{(x_1, i_1), \dots, (x_\ell, i_\ell)\} \mid x_1 \subsetneq \dots \subsetneq x_\ell \text{ in } \mathcal{P}([n]) \text{ and } (i_1 \prec \dots \prec i_\ell) \in \mathcal{G}_\ell\}.$$

By construction, a validly colored subset of $\mathcal{P}([n])$ can be viewed as an independent set in $\mathcal{A}$, though we remark that this is not a 1-1 correspondence. Also, there is a natural 1-1 correspondence between templates T and subsets of $v(\mathcal{A})$, where we assign to a template T the subset of all (x, c) with $c \in T(x)$. By a slight abuse of notation we shall sometimes view T as a subset of vertices in $v(\mathcal{A})$ and sometimes view T as the induced sub-hypergraph

$$T = \mathcal{A}|_T \subset \mathcal{A}.$$

The notion of order of the sub-hypergraph associated to T is related to the notion of weight of T by

$$|T| = |v(\mathcal{A}|_T)| = \Theta(\omega(\overline{\beta}, T)),$$

i.e. it is within a constant factor of $\omega(\overline{\beta}, T)$.

Our desired set of hypergraph containers will correspond to a family of templates which efficiently contains validly colored subsets of $\mathcal{P}([n])$.

Lemma 2.7. Suppose that $\mathcal{G}_k$ contains all colored chains of length k. For every $\alpha > 0$ there exists a $\delta > 0$ such that the following holds. Let $n \in \mathbb{N}$ and suppose that T is a template of $\mathcal{P}([n])$ supported on sets of size between $\frac{n}{3}$ and $\frac{2n}{3}$ such that $\omega(\overline{\beta}, T) \geq (\omega_{crit}(\overline{\beta}) + \alpha)\binom{n}{n/2}$. Then there exists $2 \leq \ell \leq k$ and there exists an l-uniform sub-hypergraph $\mathcal{H}_\ell$ of T such that

$$e(\mathcal{H}_\ell) \geq \delta^\ell n^{\ell-1}\binom{n}{n/2}, \text{ and}$$
$$\Delta_j(\mathcal{H}_\ell) \leq (\delta n)^{\ell-j}$$

for $1 \leq j \leq \ell$.

Corollary 2.8. For every $\alpha \in (0,1)$ there exists an $\epsilon > 0$ such that the following holds. Let $n \in \mathbb{N}$ and suppose that T is a template of $\mathcal{P}([n])$ supported on sets of size between $\frac{n}{3}$ and $\frac{2n}{3}$ such that $\omega(\overline{\beta}, T) \geq (\omega_{crit}(\overline{\beta}) + \alpha)\binom{n}{n/2}$. Then there exists a family $\mathcal{C}$ of sub-templates of T such that

(1) For every validly colored subset I with respect to $\mathcal{G}$ contained in T, there is a $T' \in \mathcal{C}$ such that I is contained in T'.
(2) We have $|\mathcal{C}| \leq e^{O(1)\frac{\log n}{n}|T|}$ for some constant $O(1)$ independent of α.
(3) We have $|T'| \leq (1-\epsilon)|T|$ for every $T' \in \mathcal{C}$.

Proof of Corollary 2.8 assuming Lemma 2.7. It suffices to prove this when $\mathcal{G}_k$ contains all colored chains of length k (as we can always augment $\mathcal{G}$ with all colored chains of length km without changing the valid configurations). Partition $T \subset v(\mathcal{A})$ into sets $T_0 \cup T_1 \cup \ldots \cup T_r$ for some $r \geq 1$ such that

$$\omega(\overline{\beta}, T_0) < (\omega_{crit}(\overline{\beta}) + \alpha)\binom{n}{n/2}, \text{ and}$$
$$\omega(\overline{\beta}, T_i) = (\omega_{crit}(\overline{\beta}) + \alpha)\binom{n}{n/2} + O(1).$$

By Lemma 2.7, there exists $\delta = \delta(\alpha)$ such that for each of $1 \leq i \leq r$ there exists an ℓ_i-uniform sub-hypergraph $\mathcal{H}^i = \mathcal{H}^i_{\ell_i}$ of the templates T_i for some $2 \leq \ell_i \leq k$ with the property that

$$e(\mathcal{H}^i) \geq \delta^{\ell_i} n^{\ell_i-1}\binom{n}{n/2}, \text{ and}$$
$$\Delta_j(\mathcal{H}^i) \leq (\delta n)^{\ell_i-j}, \text{ for all } 1 \leq j \leq \ell_i.$$

Let $2 \leq \ell \leq k$ be the most frequent uniformity. Construct the ℓ-uniform sub-hypergraph $\mathcal{H}$ of T with $v(\mathcal{H}) = v(T)$ and

$$e(\mathcal{H}) \coloneqq \bigsqcup_{\ell_i=\ell} e(\mathcal{H}^i).$$

By construction we have

$$|e(\mathcal{H})| \geq \frac{r}{k}\delta^\ell n^{\ell-1}\binom{n}{n/2}$$
$$|v(\mathcal{H})| = |v(T)| = \sum_{i=0}^{r} |v(T_i)| = O(1)\sum_{i=0}^{r} \omega(\overline{\beta}, T_i) = O(1)r\binom{n}{n/2}$$
$$\Delta_j(\mathcal{H}) \leq (\delta n)^{\ell-j} \text{ for all } 1 \leq j \leq \ell.$$

Set $\tau = \frac{1}{n}$ and $c = O(1)k\delta^{-k}$, and apply Lemma 2.6 to the l-uniform hypergraph $\mathcal{H}$ to obtain the following. There exists ϵ depending only on c, k and there exists a collection $\mathcal{C}$ of subtemplates T' of T, and a function $f\colon \mathcal{P}(v(T)) \to \mathcal{C}$ such that

(1) For every $I \in \mathcal{I}(T)$, there exists $F \subset I$ with $|F| \leq \frac{k}{n}|T|$ and $I \subset F \cup f(F)$
(2) $|T'| \leq (1-\epsilon)|T|$ for every $T' \in \mathcal{C}$.

Set $\mathcal{C}' \coloneqq \{F \cup f(F) \;:\; F \in \mathcal{C}\}$ and note that

(1) For every $I \in \mathcal{I}(T)$, there exists $T' \in \mathcal{C}'$ such that $I \subset T'$
(2) $|\mathcal{C}'| \leq \frac{k}{n}|T|\binom{|T|}{\frac{k}{n}|T|} = e^{O(1)\frac{\log(n)}{n}|T|}$.
(3) $|T'| \leq (1-\epsilon+o(1))|T|$ for every $T' \in \mathcal{C}'$.

□

Proof of upper bound of Theorem 2.2 assuming Corollary 2.8. Let $\mathcal{P}([n])'$ be all vertices in $x \in \mathcal{P}([n])$ with $|x| \in (\frac{n}{3}, \frac{2n}{3})$, and let $\Lambda'(\mathcal{G}, n)$ be the validly colored subsets of $\mathcal{P}([n])'$. We have

$$\mu(\overline{\beta}, \Lambda(\mathcal{G}, n)) \leq \left(\prod_{x \notin \mathcal{P}([n])'} (1 + \sum_{i=1}^{m} \beta_i)\right) \mu(\overline{\beta}, \Lambda'(\mathcal{G}, n)) = e^{o(1)\binom{n}{n/2}} \mu(\overline{\beta}, \Lambda'(\mathcal{G}, n)).$$

Therefore it is enough to prove that for every $\alpha > 0$ we have

$$\mu(\overline{\beta}, \Lambda'(\mathcal{G}, n)) \leq e^{(\omega_{crit}(\overline{\beta})+\alpha+o(1))\binom{n}{n/2}}.$$

Fix a threshold value $1 > \alpha > 0$. Starting with $\mathcal{A}|_{\mathcal{P}([n])' \times \{1,\ldots,m\}}$, we iteratively apply Corollary 2.8 until we obtain a family $\mathcal{C}$ of subtemplates T with $\omega(\overline{\beta}, T) \leq (\omega_{crit}(\overline{\beta})+\alpha)\binom{n}{n/2}$. This is encoded by a branching process where a template T with $\omega(\overline{\beta}, T) \geq (\omega_{crit}(\overline{\beta})+\alpha)\binom{n}{n/2}$ splits into subtemplates $T_1, T_2, \ldots, T_s$ such that each validly colored subset contained in T is contained in some T_i. By Corollary 2.8, there exists an $\epsilon = \epsilon(\alpha) > 0$ such that we have $s \leq e^{O(1)\frac{\log(n)}{n}|T|}$, and $|T_i| \leq (1-\epsilon)|T|$.

Because our initial set has size at most $m2^n$, and the size of the templates decreases by a factor of $(1-\epsilon)$ each iteration, each template at level i in this branching process splits into at most $e^{O(1)\frac{\log(n)}{n}(1-\epsilon)^i m2^n}$ other templates. Therefore, the final collection $\mathcal{C}$ of templates has cardinality bounded above by

$$|\mathcal{C}| \leq \prod_{i=0}^{\infty} e^{O(1)\frac{\log(n)}{n}(1-\epsilon)^i m2^n} = e^{O(1)\frac{log(n)}{n}\epsilon^{-1} m2^n} = e^{o(1)\binom{n}{n/2}}.$$

Note that each set in $\Lambda'(\mathcal{G},n)$ is contained in some $T\in\mathcal{C}$. Therefore, letting Λ_T be the collection of colored subsets contained in T we have by Proposition 2.4

$$\mu(\overline{\beta},\Lambda'(\mathcal{G},n))\leq\sum_{T\in\mathcal{C}}\mu(\overline{\beta},\Lambda_T)=\sum_{T\in\mathcal{C}}e^{\omega(\overline{\beta},T)}\leq e^{o(1)\binom{n}{n/2}}e^{(\omega_{crit}(\overline{\beta})+\alpha)\binom{n}{n/2}}.$$

□

3. Balanced Supersaturation

In this section, we prove Lemma 2.7. To do this, we prove a series of technical results adapted from [CM16] for our purposes. We fix a template T for the remainder of this section, and recall that by hypothesis $\mathcal{G}_k$ contains all colored chains of length k.

Definition 3.1. Define the random variables $\mathbf{X}$ and $\mathbf{Y}$ on a uniformly chosen random maximal chain C in $\mathcal{P}([n])$. Let

$$\mathbf{X}(C)=\sum_{x\in C}\log(1+|T(x)|_{\overline{\beta}}),$$

and $\mathbf{Y}(C)$ be the total number of colored subchains of C contained in T which appear as a colored chain in $\mathcal{G}$.

Definition 3.2. For $x\in\mathcal{P}([n])$ define the constant $X^x=\log(1+|T(x)|_{\overline{\beta}})$. Define the random variable $\mathbf{Y}^x$ on a uniformly chosen random maximal length chain C in $\mathcal{P}([n])$ whose top element is x by setting $\mathbf{Y}^x(C)$ to be the number of colored subchains of C contained in T whose top element is x and appear as a colored chain in $\mathcal{G}$.

Lemma 3.3. There are constants $C_1,C_2>0$ independent of n such that the following is true. For any $\alpha\in(0,C_2)$, if T is a template with $\omega(\overline{\beta},T)\geq(\omega_{crit}(\overline{\beta})+\alpha)\binom{n}{n/2}$, then there exists a vertex $x\in\mathrm{Supp}(T)$ such that

$$\mathbb{E}\mathbf{Y}^x\geq C_1\alpha.$$

Proof. Take $C_3=\log(1+\sum_{i=1}^m\beta_i)$, $C_4=\log(1+\min(\beta_i))$ and take C_1,C_2 to be

$$C_1=C_3^{-1}C_4\frac{1}{2\omega_{crit}(\overline{\beta})}$$
$$C_2=\min\{C_3^{-1}C_1^{-1}C_4,\omega_{crit}(\overline{\beta})\}=\omega_{crit}(\overline{\beta}).$$

First note that

$$\omega_{crit}(\overline{\beta})\geq\mathbf{X}(C)-C_3\mathbf{Y}(C).$$

Indeed, while $\mathbf{X}(C)>\omega_{crit}(\overline{\beta})$ we can find a forbidden colored subchain of C contained in T. Deleting one by one vertices of C from forbidden subchains, $\mathbf{Y}(C)$ decreases each time by at least 1 and $\mathbf{X}(C)$ decreases each time by at most C_3.

Now suppose for the sake of contradiction that $\mathbb{E}\mathbf{Y}^x<C_1\alpha$ for all $x\in\mathrm{Supp}(T)$. If for any $x\in\mathrm{Supp}(T)$ we have $X^x\leq C_3\mathbb{E}\mathbf{Y}^x$, we obtain the contradiction (recalling $\alpha<C_2$)

$$\log(1+\min(\beta_i))\leq X^x\leq C_3\mathbb{E}\mathbf{Y}^x<C_3C_1\alpha<C_4=\log(1+\min(\beta_i)).$$

Hence we have $X^x>C_3\mathbb{E}Y^x$ for all $x\in\mathrm{Supp}(T)$. Writing $\mathbf{X}$ and $\mathbf{Y}$ as a sum of indicator functions and using linearity of expectation we have $\mathbb{E}(\mathbf{X})=$

$\sum_{x\in \text{Supp}(T)} \frac{1}{\binom{n}{|x|}} X^x$, and $\mathbb{E}\mathbf{Y} = \sum_{x\in \text{Supp}(T)} \frac{1}{\binom{n}{|x|}} \mathbb{E}\mathbf{Y}^x$. Thus we obtain the contradiction

$$\begin{aligned}
\omega_{crit}(\overline{\beta}) \geq \mathbb{E}(\mathbf{X} - C_3\mathbf{Y}) &= \sum_{x\in \text{Supp}(T)} \frac{1}{\binom{n}{|x|}} (X^x - C_3\mathbb{E}\mathbf{Y}^x) \\
&\geq \sum_{x\in \text{Supp}(T)} \frac{1}{\binom{n}{n/2}} (X^x - C_3\mathbb{E}\mathbf{Y}^x) \geq \frac{\omega(\overline{\beta}, T)}{\binom{n}{n/2}} - \frac{|\operatorname{Supp}(T)|}{\binom{n}{n/2}} C_3C_1\alpha \\
&\geq \frac{\omega(\overline{\beta}, T)}{\binom{n}{n/2}} (1 - C_4^{-1}C_3C_1\alpha) \geq (\omega_{crit}(\overline{\beta}) + \alpha)(1 - \frac{1}{2\omega_{crit}(\overline{\beta})}\alpha) \\
&= \omega_{crit}(\overline{\beta}) + \frac{1}{2}\alpha(1 - \frac{\alpha}{\omega_{crit}(\overline{\beta})}) \\
&> \omega_{crit}(\overline{\beta}).
\end{aligned}$$

□

The following lemma, inspired by a corresponding lemma from [CM16] (adapted from an argument of [DGS15]), gives us very good control over the number of colored chains below a given vertex. It is surprising that given our non-transitive family $\mathcal{G}$ of forbidden chains that we still retain such excellent control.

Lemma 3.4. There is a constant $Q \geq 0$ independent of n such that the following is true. For any $x \in X$, $i \leq k$, and C a chain of maximal length whose top element is x, let $\mathbf{Z}^x_{c_1\succ\cdots\succ c_i}$ be the random variable such that $\mathbf{Z}^x_{c_1\succ\cdots\succ c_i}(C)$ is the number of colored subchains of C of length i contained in T, whose top element is x and is colored $c_1 \succ \cdots \succ c_i$. Then

$$\mathbf{Z}^x_{c_1\succ\cdots\succ c_i}(C) \leq Q + \mathbf{Y}^x(C).$$

Proof. Recall that $\mathcal{G}_k$ contains all chains of length k. Therefore, for $i = k$ any $Q \geq 0$ works. If $i < k$, it is enough to ensure that Q satisfies

$$\mathbf{Z}^x_{c_1\succ\cdots\succ c_i}(C) \leq Q + \sum_{j=1}^{m} \mathbf{Z}^x_{c_1\succ \underbrace{j\succ\cdots\succ j}_{k-1}}(C).$$

Indeed, if we can show this then by using the trivial bound

$$Q + \sum_{j=1}^{m} \mathbf{Z}^x_{c_1\succ \underbrace{j\succ\cdots\succ j}_{k-1}}(C) \leq Q + \mathbf{Y}^x(C),$$

the conclusion follows immediately.

If $c_1 \notin T(x)$ then the result is trivially true for any choice of $Q \geq 0$, so we assume that c_1 is a valid choice at x. Denoting $s = |C \cap \operatorname{Supp}(T)|$, we have the trival bounds

$$\mathbf{Z}^x_{c_1\succ\cdots\succ c_i}(C) \leq (s-1)^{i-1} \text{ and } \sum_{j=1}^{m} \mathbf{Z}^x_{c_1\succ \underbrace{j\succ\cdots\succ j}_{k-1}}(C) \geq \binom{\frac{s-1}{m}}{k-1},$$

where the second bound follows by observing that the most frequent color on C appears at least $\frac{s-1}{m}$ times.

Thus it suffices to take Q such that

$$(s-1)^{i-1} \leq Q + \binom{\frac{s-1}{m}}{k-1}$$

for every $1 \leq i < k$, and every $s \geq 1$. □

Proof of Lemma 2.7. We build an auxiliary sub-hypergraph $\mathcal{H}$ of T one edge at a time, ensuring with each new edge that $\Delta_j(\mathcal{H}_\ell) \leq (\delta n)^{\ell-j}$ holds for all ℓ and $1 \leq j \leq \ell$, where $\mathcal{H}_\ell$ is the set of ℓ-edges of $\mathcal{H}$, until for some ℓ we have $e(\mathcal{H}_\ell) \geq \delta^{\ell-1} n^\ell \binom{n}{n/2}$, and then we output $\mathcal{H}_\ell$.[1] In particular we assume that at the current stage $e(\mathcal{H}_\ell) < \delta^{\ell-1} n^\ell \binom{n}{n/2}$ for all ℓ. Note that given a colored chain B contained in T that also appears in $\mathcal{G}$, if it cannot be added to $\mathcal{H}$, then there exists a colored subchain B' satisfying $d_{\mathcal{H}_\ell}(B') = (\delta n)^{\ell-|B'|}$ for some ℓ, so adding B to $\mathcal{H}$ would violate the codegree condition. We will implicitly find such a colored chain B which we can add to $\mathcal{H}$ by constructing it one vertex at a time from the top down, ensuring that no codegree condition among the subsets of B is violated at each step. The following claim shows that there are very few ways of extending a "good" B to a "bad" B with the addition of a vertex.

Claim 3.5. Given an i-chain $x_1 \supsetneq \ldots \supsetneq x_i$ with x_j colored by $c_j \in T(x_j)$, then there are at most $O(1)\delta n$ choices for (x_{i+1}, c_{i+1}) with $x_i \supsetneq x_{i+1}$ and $c_{i+1} \in T(x_{i+1})$ such that there exists some $2 \leq \ell \leq k$, and *nonempty* $A \subset \{(x_1, c_1), \ldots, (x_i, c_i)\}$ with $|A| \leq \ell - 1$ such that $d_{\mathcal{H}_\ell}(A \cup \{(x_{i+1}, c_{i+1})\}) = (\delta n)^{\ell-(|A|+1)}$.

Proof. Fix some $A \subset \{(x_1, c_1), \ldots, (x_i, c_i)\}$ and $2 \leq \ell \leq k$, and let $\mathcal{B}$ be the set of all (x_{i+1}, c_{i+1}) with $x_{i+1} \subsetneq x_i$ and $c_{i+1} \in T(x_{i+1})$ and $d_{\mathcal{H}_\ell}(A \cup \{(x_{i+1}, c_{i+1})\}) = (\delta n)^{\ell-(|A|+1)}$. The multiset union

$$\bigsqcup_{(x_{i+1}, c_{i+1}) \in \mathcal{B}} \text{hyperedges of } \mathcal{H}_\ell \text{ containing } A \cup \{(x_{i+1}, c_{i+1})\}$$

has size $|\mathcal{B}|(\delta n)^{\ell-(|A|+1)}$, and each edge appears at most k times. Ignoring repeats, this is a collection of hyperedges of $\mathcal{H}_\ell$ containing A, so has at most $\Delta_{|A|}(\mathcal{H}_\ell) \leq (\delta n)^{\ell-|A|}$ distinct elements. Therefore, we conclude that $|\mathcal{B}| \leq k\delta n$. Summing over all choices of A, ℓ we obtain the desired result. □

We remark that the claim (via the nonemptiness condition on A) does not take into account the possibility that the new colored vertex v added to B violates the $\Delta_1(\mathcal{H}_\ell)$ condition for the singleton $\{v\}$. However, we will explicitly deal with this possibility by disregarding such colored vertices.

We continue the proof of Lemma 2.7 along the lines of [CM16], which we include for completeness (and rephrase in terms of random variables for convenience). By a double counting argument, the number of colored vertices v with $d_{\mathcal{H}_\ell}(\{v\}) = (\delta n)^{\ell-1}$ is at most $\ell e(\mathcal{H}_\ell)/(\delta n)^{\ell-1} \leq \ell\delta\binom{n}{n/2}$. Omitting these colored vertices, and assuming we take $\delta < \alpha/(2k\log(1+\sum_{i=1}^m \beta_i))$, we obtain a template T' with $\omega(\overline{\beta}, T') \geq (\omega_{crit}(\overline{\beta}) + \frac{\alpha}{2})\binom{n}{n/2}$. We will now show there is a colored chain contained in T' which we can add to $\mathcal{H}$. For the remainder of the proof we take all random variables with respect to T' rather than T.

[1]This is analogous to [CM16], except since they worked in a single uniformity they could directly construct their final hypergraph without using an auxiliary hypergraph.

By Lemma 3.3 applied to T', we can take $x \in \mathrm{Supp}(T')$ minimal such that $\mathbb{E}\mathbf{Y}^x \geq C_1\frac{\alpha}{2}$. Then for $x' \subsetneq x$, and $c_1, \ldots, c_j$ with $j \leq k$ by Lemma 3.4 and the minimality of x we have

$$\mathbb{E}\mathbf{Z}^{x'}_{c_1 \succ \cdots \succ c_j} \leq Q + \mathbb{E}\mathbf{Y}^{x'} \leq Q + C_1\frac{\alpha}{2}$$

and

$$\mathbb{E}\mathbf{Z}^{x}_{c_1 \succ \cdots \succ c_j} \leq Q + \mathbb{E}\mathbf{Y}^{x}.$$

Our goal is to find a colored chain with x as its top element contained in T' that we can add to $\mathcal{H}$ without violating any of the codegree conditions. To do this, we write the random variable $\mathbf{Y}^x = \mathbf{Y}^x_{bad} + \mathbf{Y}^x_{good}$ where $\mathbf{Y}^x_{bad}$ only counts colored chains of $\mathcal{G}$ contained in T' we are not allowed to add to $\mathcal{H}$. It suffices to prove the upper bound

$$\mathbb{E}\mathbf{Y}^x_{bad} \leq \sum_{\ell} \sum_{c_1,\ldots,c_l} \sum_{1 \leq i \leq \ell} \mathbb{E}\mathbf{Z}^x_{c_1 \succ \cdots \succ c_i} \cdot \left(O(1)\delta n \frac{1}{n/3}\right) \cdot \max_{x' \subsetneq x} \mathbb{E}\mathbf{Z}^{x'}_{c_{i+1} \succ \cdots \succ c_\ell}.$$

Indeed, by the above the right hand side is bounded above by $(Q+\mathbb{E}\mathbf{Y}^x)(O(1)\delta)(Q+C_1\frac{\alpha}{2})$, and by choosing δ sufficiently small in terms of α and the absolute constant Q (independent of $\mathbb{E}\mathbf{Y}^x$), we can guarantee this is strictly less than $\mathbb{E}\mathbf{Y}^x$ (using the fact that $\mathbb{E}\mathbf{Y}^x \geq C_1\frac{\alpha}{2}$). Therefore $\mathbf{Y}^x_{good}$ is not identically zero and we can find a new hyperedge to add to $\mathcal{H}$.

To do this we first similarly split $\mathbf{Z}^x_{c_1 \succ \cdots \succ c_\ell} = \mathbf{Z}^x_{c_1 \succ \cdots \succ c_\ell, bad} + \mathbf{Z}^x_{c_1 \succ \cdots \succ c_\ell, good}$, and write

$$\mathbf{Y}^x_{bad}(C) = \sum_{\ell} \sum_{(c_1 \succ \cdots \succ c_\ell) \in \mathcal{G}_\ell} \mathbf{Z}^x_{c_1 \succ \cdots \succ c_\ell, bad}(C).$$

Next, we upper bound

$$\mathbf{Z}^x_{c_1 \succ \cdots \succ c_\ell, bad}(C) \leq \sum_{i} \sum_{x_i \subsetneq x} \sum_{x_{i+1} \subsetneq x_i} \mathbf{Z}^x_{c_1 \succ \cdots \succ c_\ell, bad, i, x_i, x_{i+1}}(C)$$

where $\mathbf{Z}^x_{c_1 \succ \cdots \succ c_\ell, bad, i, x_i, x_{i+1}}$ counts those colored subchains of C contained in T', whose top element is x, colored $c_1 \succ \cdots \succ c_\ell$, such that the i'th and $i+1$'st elements from the top are precisely at the locations x_i, x_{i+1} respectively, and furthermore that x_{i+1} along with some subset of the colored elements of the chain above it violate some codegree condition.

To bound the expectation of the right hand side of this triple sum, we first note that

$$\begin{aligned}\mathbb{E}\mathbf{Z}^x_{c_1 \succ \cdots \succ c_\ell, bad, i, x_i, x_{i+1}} &= (\mathbb{E}\mathbf{Z}^{x_{i+1}}_{c_{i+1} \succ \cdots \succ c_\ell})\mathbb{E}\mathbf{Z}^x_{c_1 \succ \cdots \succ c_{i+1}, bad, i, x_i, x_{i+1}} \\ &\leq (\max_{x' \subsetneq x} \mathbb{E}\mathbf{Z}^{x'}_{c_{i+1} \succ \cdots \succ c_\ell}) \cdot \mathbb{E}\mathbf{Z}^x_{c_1 \succ \cdots \succ c_{i+1}, bad, i, x_i, x_{i+1}}\end{aligned}$$

By Claim 3.5, we now have

$$\sum_{x_{i+1} \subsetneq x_i} \mathbb{E}\mathbf{Z}^x_{c_1 \succ \cdots \succ c_{i+1}, bad, i, x_i, x_{i+1}} \leq O(1)\delta n \frac{1}{n/3} \mathbb{E}\mathbf{Z}^x_{c_1 \succ \cdots \succ c_i, x_i}$$

where $\mathbf{Z}^x_{c_1 \succ \cdots \succ c_i, x_i}(C)$ counts those colored subchains of C contained in T', whose top element is x, colored $c_1 \succ \cdots \succ c_i$, such that the bottom element is x_i.

Finally, note that

$$\sum_{x_i \subsetneq x} \mathbf{Z}^x_{c_1 \succ \cdots \succ c_i, x_i}(C) = \mathbf{Z}^x_{c_1 \succ \cdots \succ c_i}(C).$$

Putting this all together now yields the desired inequality.

□

References

[BMS15] József Balogh, Robert Morris, and Wojciech Samotij. Independent sets in hypergraphs. *J. Amer. Math. Soc.*, 28(3):669–709, 2015.

[BMS18] József Balogh, Robert Morris, and Wojciech Samotij. The method of hypergraph containers, 2018.

[BMT14] József Balogh, Richard Mycroft, and Andrew Treglown. A random version of Sperner's theorem. *J. Combin. Theory Ser. A*, 128:104–110, 2014.

[BTW16] József Balogh, Andrew Treglown, and Adam Zsolt Wagner. Applications of graph containers in the Boolean lattice. *Random Structures Algorithms*, 49(4):845–872, 2016.

[CM16] Maurício Collares and Robert Morris. Maximum-size antichains in random set-systems. *Random Structures Algorithms*, 49(2):308–321, 2016.

[DGS15] Shagnik Das, Wenying Gan, and Benny Sudakov. Sperner's theorem and a problem of Erdős, Katona and Kleitman. *Combin. Probab. Comput.*, 24(4):585–608, 2015.

[FOU19] Victor Falgas-Ravry, Kelly O'Connell, and Andrew Uzzell. Multicolor containers, extremal entropy, and counting. *Random Structures & Algorithms*, 54(4):676–720, 2019.

[Kle69] Daniel Kleitman. On Dedekind's problem: The number of monotone Boolean functions. *Proc. Amer. Math. Soc.*, 21:677–682, 1969.

[ST15] David Saxton and Andrew Thomason. Hypergraph containers. *Invent. Math.*, 201(3):925–992, 2015.