THE OPERATIONAL PERSPECTIVE

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Advances in Proof Theory
In honor of Gerhard Jäger’s 60th birthday
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Operationally Based Axiomatic Programs

• The Explicit Mathematics Program
• The Unfolding Program
• A Logic for Mathematical Practice
• Operational Set Theory (OST)
Foundations of Explicit Mathematics

• Book in progress with Gerhard Jäger and Thomas Strahm, with the assistance of Ulrik Buchholtz

• An online bibliography
The Unfolding Program

• Open-ended Axiomatic Schemata; language not fixed in advance

• Examples in Logic, Arithmetic, Analysis, Set Theory

• The general concept of unfolding explained within an operational framework
Aim of the Unfolding Program

• S an open-ended schematic axiom system

• Which operations on individuals--and which on predicates--and what principles concerning them ought to be accepted once one has accepted the operations and principles of S?
Results on (Full) Unfolding

• Non-Finitist Arithmetic (NFA);
  $|\mathcal{U}(\text{NFA})| = \Gamma_0$

• Finitist Arithmetic (FA):
  $\mathcal{U}(\text{FA}) \equiv \text{PRA}, \mathcal{U}(\text{FA} + \text{BR}) \equiv \text{PA}$

• (Feferman and Strahm 2000, 2010)
Unfolding of ID₁

• \(|U(ID₁)| = ψ(Γ_{Ω+1})|
  (U. Buchholtz 2013)

• **Note:** \(ψ(Γ_{Ω+1})\) is to \(ψ(ε_{Ω+1})\) as \(Γ₀\) is to \(ε₀\).
Problems for Unfolding to Pursue

• Unfolding of analysis
• Unfolding of KP + Pow
• Unfolding of set theory
Indescribable Cardinals and Admissible Analogues Revisited

• **Aim**: To have a straightforward and principled transfer of the notions of indescribable cardinals from set theory to admissible ordinals.

• A new proposal and several conjectures, suggested at the end of the OST paper.

• **NB**: Not within OST
Aczel and Richter
Pioneering Work

• Aczel and Richter [A-R] (1972)

• In set theory, assume $\kappa$ regular $> \omega$.

• Let $f, g: \kappa \to \kappa$; $F(f) = g$ type 2 over $\kappa$. 
• F is bounded $\iff (\forall f: \kappa \to \kappa)(\forall \xi < \kappa)$
  [ F(f)(\xi) is det. by $< \kappa$ values of f ]

• $\alpha$ is a witness for F $\iff (\forall f: \kappa \to \kappa)$
  [f :$\alpha \to \alpha \Rightarrow F(f): \alpha \to \alpha$]

• $\kappa$ is 2-regular iff every bounded F has a witness.
• Notions of bounded, witness, \( n \)-regular for \( n > 2 \) are “defined in a similar spirit”, but never published.

• **Theorem 1.** \( \kappa \) is \( n+1 \)-regular iff \( \kappa \) is strongly \( \Pi^1_n \)-indescribable.

• Proved only for \( n = 1 \) in [R-A](1974).
Admissible analogues:

Assume \( \kappa \) admissible \( > \omega \)

\( \kappa \) is \( n \)-admissible, obtained by replacing ‘bounded’ in the defn. of \( n \)-regular by ‘recursive’, functions by their Gödel indices, and functionals by recursive functions applied to such indices.
• **Theorem 2.** $\kappa$ is $n$-admissible iff $\kappa$ is $\Pi^0_{n+1}$ reflecting.

• Proved only for $n = 2$ in [R-A](1974).

• **Proposed:**
  Least $\Pi^0_{n+2}$-reflecting ordinal $\sim$ least [strongly] $\Pi^1_n$-indescribable cardinal.
A Proposed New Approach

- Directly lift to card’s and admissible ord’s notions of continuous functionals of finite type from o.r.t.
- Kleene (1959), Kreisel (1959)
- Deal only with objects of pure type $n$.
- $\kappa^{(0)} = \kappa; \kappa^{(n+1)} = \text{all } F^{(n+1)}: \kappa^{(n)} \rightarrow \kappa$. 
“Sequence Numbers” in Set Theory

• Assume $\kappa$ a strongly inaccessible cardinal.

• Let $\kappa^{<\kappa}$ = all sequences $s: \alpha \rightarrow \kappa$ for arbitrary $\alpha < \kappa$.

• Fix $\pi: \kappa^{<\kappa} \rightarrow \kappa$, one-one and onto; so $\pi(g\upharpoonright \alpha)$ is an ordinal that codes $g\upharpoonright \alpha$. 
Continuous Functionals and Their Associates

• Inductive definition of $F \in C^{(n)}$, and of $f$ is an associate of $F$, where $f$ is of type 1:

• For $n = 1$, $f$ is an associate of $F$ iff $f = F$.

• For $F \in K^{(n+1)}$, $f$ is an associate of $F$ iff for every $G$ in $C^{(n)}$ and every associate $g$ of $G$,
Continuous Functionals and Their Associates (cont’d)

• (i) \( (\exists \alpha, \beta < \kappa)(\forall \gamma)[\alpha \leq \gamma < \kappa \Rightarrow f(\pi(g\setminus\gamma)) = \beta + 1] \), and

• (ii) \( (\forall \gamma, \beta < \kappa) \ [f(\pi(g\setminus\gamma)) = \beta + 1 \Rightarrow F(G) = \beta] \).

• \( F \) is in \( C^{(n+1)} \) iff \( F \) has some associate \( f \).
Witnesses

• For $F$ in $C^{(n)}$ and $\alpha < \kappa$, define $\alpha$ is a witness for $F$, as follows:

• For $n = 1$, and $F = f$, $\alpha$ is a witness for $F$ iff $f : \alpha \to \alpha$.

• For $F \in C^{(n+1)}$, $\alpha$ is a witness for $F$ iff $(\forall G \in C^{(n)})[\alpha$ a witness for $G \Rightarrow F(G) < \alpha ]$. 
C^{(n)}-Regularity; Conjectures

• \( \kappa \) is \( C^{(n)} \)-\( \text{reg} \) for \( n > 1 \) iff every \( F \) in \( C^{(n)} \) has some witness \( \alpha < \kappa \).

• **Conjecture 1.** For each \( n \geq 1 \), the predicate \( f \) is an associate of some \( F \) in \( C^{(n+1)} \), is definable in \( \Pi^1_n \) form.

• **Conjecture 2.** For each \( n \geq 1 \), \( \kappa \) is \( C^{(n+1)} \)-\( \text{reg} \) iff \( \kappa \) is strongly \( \Pi^1_n \)-indescribable.

• Conj-2 holds for \( n = 1 \) by [R-A] proof.
Analogues over Admissibles

• Consider admissible $\kappa > \omega$.

• For analogues in ($\kappa$-) recursion theory replace functions of type 1 by indices $\zeta$ of (total) recursive functions $\{\zeta\}$.

• But then at type 2 (and higher) we must restrict to those functions $\{\zeta\}$ that act extensionally on indices.
Effective Operations over Admissibles

• Following Kreisel (1959), define the class $E_n$ of (κ-) effective operations of type $n$, and the relation $≡_n$ by induction on $n > 0$:

  $E_1$ consists of all indices $ζ$ of recursive functions;
  $ζ ≡_1 ν$ iff for all $ξ$, $\{ζ\}(ξ) = \{ν\}(ξ)$.
Effective Operations over Admissibles (cont’d)

- \( \zeta \in E_{n+1} \iff \{\zeta\}: E_n \to \kappa \) and
- \((\forall \xi, \eta \in E_n)[ \xi \equiv_n \eta \Rightarrow \{\zeta\}(\xi) = \{\zeta\}(\eta)]\);
- \( \zeta \equiv_{n+1} \nu \iff (\forall \xi \in E_n)[\{\zeta\}(\xi) = \{\nu\}(\xi)]\).

- **Conjecture 3.** Every type \( n+1 \) effective operation is the restriction of a functional in \( C^{(n+1)} \).

- This would show why can drop the boundedness hypothesis in analogue.
Witnesses for Effective Operations

• For \( \zeta \) in \( E_1 \), \( \alpha \) is a witness for \( \zeta \) iff \( \{ \zeta \} : \alpha \rightarrow \alpha \).

• For \( \zeta \) in \( E_{n+1} \) when \( n \geq 1 \), \( \alpha \) is a witness for \( \zeta \leftrightarrow (\forall \xi \in E_n) \)
  \[ \alpha \text{ a witness for } \xi \Rightarrow \{ \zeta \}(\xi) < \alpha \].

• \( \kappa \) is \( E_n\)-admissible if each \( \zeta \) in \( E_n \) has some witness \( \alpha < \kappa \). (Equiv. to \([A, R] \) n-admiss.)
Further Work

• Settle the conjectures.

• (Scott) The partial equivalence relation approach to types in \( \lambda \)-calculus models over \( P(N) \) gives a "clean" definition of the Kleene-Kreisel hierarchy. Can this idea be generalized to \( P(\kappa) \)? [What about effective operations?]
Further Work (cont’d)

• The present approach leaves open the question as to what is the proper analogue for admissible ordinals--if any--of a cardinal $\kappa$ being $\Pi^m_n$-indescribable for $m > 1$. 
The End