Logic, Mathematics and Conceptual Structuralism

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Abstract. Conceptual structuralism is a non-realist philosophy of mathematics according to which the objects of mathematical thought are humanly conceived "ideal-world" structures. Basic conceptions of structures, such as those of the natural numbers, the continuum, and sets in the cumulative hierarchy, differ in their degree of clarity. One may speak of what is true in a given conception, but that notion of truth may be partial. Mathematics proceeds from such basic conceptions by reflective expansion and carefully reasoned argument, the last of which is analyzed in logical terms. The main questions for the role of logic here is whether there are principled demarcations on its use. It is claimed that in the case of a completely clear conception, such as that of the natural numbers, the logical notions are just those of first-order classical logic and hence that that is the appropriate vehicle for reasoning. At the other extreme, in the case of set theory, where each set is conceived of as a definite totality but the universe of "all" sets is an indefinite totality, it is proposed that the appropriate logic is semi-intuitionistic in which classical logic applies only to (set-) bounded formulas. Certain subsystems of classical set theory in which extensive parts of mathematics can be formalized are reducible to these semi-intuitionistic systems, thus justifying the *de facto* use of classical logic in mathematical practice at least to that reach.

The nature and role of logic in mathematics: three perspectives. Logic is integral to mathematics and, to the extent that that is the case, a philosophy of logic should be integral to a philosophy of mathematics. In this, as you shall see, I am guided throughout by the simple view that what logic is to provide is all those forms of reasoning that lead invariably from truths to truths. The problematic part of this is how we take the notion of truth to be given. My concerns here are almost entirely with the nature and role of logic in mathematics. In order to examine that we need to consider three perspectives: that of the working mathematician, that of the mathematical logician and that of the philosopher of mathematics.

The aim of the mathematician working in the mainstream is to establish truths about mathematical concepts by means of proofs as the principal instrument. We have to look to practice to see what is accepted as a mathematical concept and what is accepted as a proof; neither is determined formally. As to concepts, among specific ones the integer and real number systems are taken for granted, and among general ones, notions of finite and infinite sequence, set and function are ubiquitous; all else is successively explained in terms of basic ones such as these. As to proofs, even though current standards of rigor require closely reasoned arguments, most mathematicians make no explicit reference to the role of logic in them, and few of them have studied logic in any systematic way. When mathematicians consider axioms, instead it is for specific kinds of structures: groups, rings, fields, linear spaces, topological spaces, metric spaces, Hilbert spaces, categories, etc., etc. Principles of a foundational character are rarely mentioned, if at all, except on occasion for proof by contradiction and proof by induction. The least upper bound principle on bounded sequences or sets of real numbers is routinely applied without mention. Some notice is paid to applications of the Axiom of Choice. To a side of the mainstream are those mathematicians such as constructivists or semi-constructivists who reject one or another of commonly accepted principles, but even for them the developments are largely informal with little explicit attention to logic. And, except for some far outliers, what they do is still recognizable as mathematics to the mathematician in the mainstream.

Turning now to the logicians' perspective, one major aim is to model mathematical practice—ranging from the local to the global—in order to draw conclusions about its potentialities and limits. In this respect, then, mathematical logicians have their own practice; here I shall sketch it and only later take up the question how well it meets that aim. In brief: Concepts are tied down within formal languages and proofs within formal systems, while truth, be it for the mainstream or for the outliers, is explained in semantic terms. Some familiar formal systems for the mainstream are Peano Arithmetic (PA), Second-Order Arithmetic (PA^2), and Zermelo-Fraenkel set theory (ZF); Heyting Arithmetic (HA) is an example of a formal system for the margin. In their intended or "standard" interpretations, PA and HA deal specifically with the natural numbers, PA^2 deals with the natural numbers and arbitrary sets of natural numbers, while ZF deals with the sets in the cumulative hierarchy. Considering syntax only, in each case the well-formed formulas of each of these systems are generated from its atomic formulas (corresponding to the basic concepts involved) by closing under some or all of the "logical" operations of negation, conjunction, disjunction, implication, universal and existential quantification.

The case of PA^2 requires an aside; in that system the quantifiers are applied to both the first-order and second-order variables. But we must be careful to distinguish the logic of quantification over the second-order variables as it is applied formally within PA^2 from its role in second-order logic under the so-called standard interpretation. In order to distinguish systematically between the two, I shall refer to the former as *syntactic* or *formal second-order logic* and the latter as *semantic* or *interpreted second-order logic*. In its pure form over any domain for the first-order variables, semantic second-order logic takes the domain of the second-order variables to be the supposed totality of arbitrary subsets of that domain; in its applied form, the domain of first-order variables has some specified interpretation. As an applied second-order formal system, PA^2 may equally well be considered to be a *two-sorted first-order theory*; the only thing that acknowledges its intended second-order interpretation is the inclusion of the so-called Comprehension Axiom Scheme: that consists of all formulas of the form $\exists X\forall x[x \in X \leftrightarrow A(x,\ldots)]$ where A is an arbitrary formula of the language of PA^2 in which 'X' does not occur as a free variable. Construing things in that way, the formal logic of all of the above-mentioned systems may be taken to be first-order.

Now, it is a remarkable fact that *all the formal systems* that have been set up to model mathematical practice are in effect based on first-order logic, more specifically its classical system for mainstream mathematics and its intuitionistic system for constructive mathematics. (While there are formal systems that have been proposed involving extensions of first-order logic by, for example, modal operators, the purpose of such has been philosophical. These operators are not used by mathematicians as basic or defined mathematical concepts or to reason about them.) One can say more about why this is so than that it happens to be so; that is addressed below.

The third perspective to consider on the nature and role of logic in mathematics is that of the philosopher of mathematics. Here there are a multitude of positions to consider; the principal ones are logicism (and neo-logicism), "platonic" realism, constructivism, formalism, finitism, predicativism, naturalism, and structuralism.[1] Roughly speaking, in all of these except for constructivism, finitism and formalism, classical first-order logic is

[1] Most of these are surveyed in the excellent collection Shapiro (2005).

either implicitly taken for granted or explicitly accepted. In constructivism (of the three exceptions) the logic is intuitionistic, i.e. it differs from the classical one by the exclusion of the Law of Excluded Middle (LEM). According to formalism, any logic may be chosen for a formal system. In finitism, the logic is restricted to quantifier-free formulas for decidable predicates; hence it is a fragment of both classical and intuitionistic logic. At the other extreme, classical second-order logic is accepted in set-theoretic realism, and that underlies both scientific and mathematical naturalism; it is also embraced in *in re* structuralism. Modal structuralism, on the other hand, expands that via modal logic. The accord with mathematical practice is perhaps greatest with mathematical naturalism, which simply takes practice to be the given to which philosophical methodology must respond. But the structuralist philosophies take the most prominent conceptual feature of modern mathematics as their point of departure.

Conceptual structuralism. This is an ontologically non-realist philosophy of mathematics that I have long advanced; my main concern here is to elaborate the nature and role of logic within it. I have summarized this philosophy in Feferman (2009) via the following ten theses.[2]

1. The basic objects of mathematical thought exist only as mental conceptions, though the source of these conceptions lies in everyday experience in manifold ways, in the processes of counting, ordering, matching, combining, separating, and locating in space and time.

2. Theoretical mathematics has its source in the recognition that these processes are independent of the materials or objects to which they are applied and that they are potentially endlessly repeatable.

3. The basic conceptions of mathematics are of certain kinds of relatively simple ideal-world pictures that are not of objects in isolation but of structures, i.e. coherently conceived groups of objects interconnected by a few simple relations and operations. They are communicated and understood prior to any axiomatics, indeed prior to any

[2] This section is largely taken from Feferman (2009), with a slight rewording of theses 5 and 10.

systematic logical development.

4. Some significant features of these structures are elicited directly from the world-pictures that describe them, while other features may be less certain. Mathematics needs little to get started and, once started, a little bit goes a long way.

5. Basic conceptions differ in their degree of clarity or definiteness. One may speak of what is true in a given conception, but that notion of truth may be partial. Truth in full is applicable only to completely definite conceptions.

6. What is clear in a given conception is time dependent, both for the individual and historically.

7. Pure (theoretical) mathematics is a body of thought developed systematically by successive refinement and reflective expansion of basic structural conceptions.

8. The general ideas of order, succession, collection, relation, rule and operation are pre-mathematical; some implicit understanding of them is necessary to the understanding of mathematics.

9. The general idea of property is pre-logical; some implicit understanding of that and of the logical particles is also a prerequisite to the understanding of mathematics. The reasoning of mathematics is in principle logical, but in practice relies to a considerable extent on various forms of intuition in order to arrive at understanding and conviction.

10. The objectivity of mathematics lies in its stability and coherence under repeated communication, critical scrutiny and expansion by many individuals often working independently of each other. Incoherent concepts, or ones that fail to withstand critical examination or lead to conflicting conclusions are eventually filtered out from mathematics. The objectivity of mathematics is a special case of intersubjective objectivity that is ubiquitous in social reality.

Two basic structural conceptions. These theses are illustrated in Feferman (2009) by the conception of the structure of the positive integers on the one hand and by several conceptions of the continuum on the other. Since our main purpose here is to elaborate the nature and role of logic in such structural conceptions, it is easiest to review here what I wrote there, except that I shall limit myself to the set-theoretical conception of the continuum in the latter case.

The most primitive mathematical conception is that of the positive integer sequence as represented by the tallies: |, ||, |||, ... From the structural point of view, our conception is that of a structure (N^+, 1, Sc, <), where N^+ is generated from the initial unit 1 by closure under the successor operation Sc, and $m < n$ if m precedes n in the generation procedure. Certain facts about this structure (if one formulates them explicitly at all), are evident: that < is a total ordering of N^+ for which 1 is the least element, and that $m < n$ implies $Sc(m) < Sc(n)$. Reflecting on a given structure may lead us to elaborate it by adjoining further relations and operations and to expand basic principles accordingly. For example, in the case of N^+, thinking of concatenation of tallies immediately leads us to the operation of addition, $m + n$, and that leads us to $m \times n$ as "m added to itself n times". The basic properties of the + and × operations such as commutativity, associativity, distributivity, and cancellation are initially recognized only implicitly. We may then go on to introduce more distinctively mathematical notions such as the relations of divisibility and congruence and the property of being a prime number. In this language, a wealth of interesting mathematical statements can already be formulated and investigated as to their truth or falsity, for example, that there are infinitely many twin prime numbers, that there are no odd perfect numbers, Goldbach's conjecture, and so on.

The conception of the structure (N^+, 1, Sc, <, +, ×) is so intuitively clear that (again implicitly, at least) there is no question in the minds of mathematicians as to the definite meaning of such statements and the assertion that they are true or false, independently of whether we can establish them in one way or the other. (For example, it is an open problem whether Goldbach's conjecture is true.) In other words, realism in truth values is accepted for statements about this structure, and the application of classical logic in reasoning about such statements is automatically legitimized. Despite the "subjective" source of the positive integer structure in the collective human understanding, it lies in the domain of objective concepts and there is no reason to restrict oneself to intuitionistic logic on subjectivist grounds. Further reflection on the structure of positive integers with the aim to simplify calculations and algebraic operations and laws leads directly to its extension to the structure of natural numbers (N, 0, Sc, <, +, ×), and then the usual structures for the integers Z and the rational numbers Q. The latter are relatively refined conceptions, not basic ones, but we are no less clear in our dealings with them than for the basic conceptions of N^+.

At a further stage of reflection we may recognize the least number principle for the natural numbers, namely if $P(n)$ is any well-defined property of members of N and there is some n such that $P(n)$ holds then there is a least such n. More advanced reflection leads to general principles of proof by induction and definition by recursion on N. Furthermore, the general scheme of induction,

$$P(0) \wedge \forall n[P(n) \rightarrow P(Sc(n))] \rightarrow \forall n P(\mathrm{n}),$$

is taken to be open-ended in the sense that it is accepted for any definite property P of natural numbers that one meets in the process of doing mathematics, no matter what the subject matter and what the notions used in the formulation of P. The question—What is a definite property?—requires in each instance the mathematician's judgment. For example, the property, "n is an odd perfect number," is definite, while "n is a feasibly computable number" is not, nor is "n is the number of grains of sand in a heap."

Turning now to the continuum, in Feferman (2009) I isolated several conceptions of it ranging from the straight line in Euclidean geometry through the system of real numbers to the set of all subsets of the natural numbers. The reason that these are all commonly referred to as *the continuum* is that they have the same cardinal number; however, that ignores essential conceptual differences. For our purposes here, it is sufficient to concentrate on the last of these concepts. The general idea of set or collection of objects is of course ancient, but it only emerged as an object of mathematical study at the hands of Georg Cantor in the 1870s. Given the idea of an arbitrary set X of elements of any given set D, considered independently of how membership in X may be defined, we write

$S(D)$ for the conception of the *totality of all subsets* X of D. Then the continuum in the set-theoretical sense is simply that of the set $S(N)$ of all subsets of N. This may be regarded as a two-sorted structure, $(N, S(N), \in)$, where $\in$ is the relation of membership of natural numbers to sets of natural numbers. Two principles are evident for this conception, using letters 'X', 'Y' to range over $S(N)$ and 'n' to range over N.

I. *Extensionality* $\quad \forall X \forall Y[\forall n(n \in X \leftrightarrow n \in Y) \rightarrow X = Y]$

II. *Comprehension* $\quad$ For any definite property P(n) of members of N,

$$\exists X \forall n[n \in X \leftrightarrow P(n)].$$

What is problematic here for conceptual structuralism is the meaning of 'all' in the description of $S(N)$ as comprising *all* subsets of N. According to the usual set-theoretical view, $S(N)$ is a definite totality, so that quantification over it is well-determined and may be used to express definite properties P. But again that requires on the face of it a realist ontology and in that respect goes beyond conceptual structuralism. So if we do not subscribe to that, we may want to treat $S(N)$ as indefinite in the sense that it is open-ended. Of course this is not to deny that we recognize many properties P as definite such as—to begin with—all those given by first-order formulas in the language of the structure $(N, 0, Sc, <, +, \times)$ (i.e. those that are ordinarily referred to as the arithmetical properties); thence any sets defined by such properties are recognized to belong to $S(N)$.

Incidentally, even from this perspective one can establish *categoricity* of the Extensionality and Comprehension principles for the structure $(N, S(N), \in)$ relative to N in a straightforward way as follows. Suppose given another structure $(N, S'(N), \in')$, satisfying the principles I and II, using set variables 'X'' and 'Y'' ranging over $S'(N)$. Given an X in $S(N)$, let $P(n)$ be the definite property, $n \in X$. Using Comprehension for the structure $(N, S'(N), \in')$, one obtains existence of an X' such that for all n in N, $n \in X$ iff $n \in X'$; then X' is unique by Extensionality. This gives a one-one map of $S(N)$ into $S'(N)$ preserving N and the membership relation; it is seen to be an onto map by reversing the argument. This is to be compared with the standard set-theoretical view of categoricity results as exemplified, for example, in Shapiro (1997) and Isaacson (2011). According to that view, the subject matter of mathematics is structures, and the *mère* structures of mathematics such as the natural numbers, the continuum (in one of its various guises), and suitable initial segments of the cumulative hierarchy of sets are characterized by axioms in full second-order logic; that is, any two structures satisfying the same such axioms are isomorphic.[3] On that account, the proofs of categoricity in one way or another then appeal *prima facie* to the presumed totality of arbitrary subsets of any given set.[4]

Even if the definiteness of $S(N)$ is open to question as above, we can certainly conceive of a world in which $S(N)$ is a definite totality and quantification over it is well-determined; in that ideal world, one may take for the property P in the above Comprehension Principle any formula of full second-order logic over the language of arithmetic. Then a number of theorems can be drawn as consequences in the corresponding system PA^2, including purely arithmetical theorems. Since the truth definition for arithmetic can be expressed within PA^2 and transfinite induction can be proved in it for very large recursive well-orderings, PA^2 goes in strength far beyond PA

[3] Those who subscribe to this set-theoretical view of the categoricity results may differ on whether the existence of the structures in question follows from their uniqueness up to isomorphism. Shapiro (1997), for example, is careful to note repeatedly that it does not, while Isaacson (2011) apparently asserts that it does (cf., e.g., op. cit. p.3). In any case, it is of course not a logical consequence.

[4]In general, proofs of categoricity within formal systems of second-order logic can be analyzed to see just what parts of the usual impredicative comprehension axiom scheme are needed for them. In the case of the natural number structure, however, it may be shown that there is no essential dependence at all, in contrast to standard proofs. Namely, Simpson and Yokoyama (2012) demonstrate the categoricity of the natural numbers (as axiomatized with the induction axiom in second-order form) within the very weak subsystem WKL_0 of PA^2 that is known to be conservative over PRA (Primitive Recursive Arithmetic). By comparison, it is sketched in Feferman (2013) how to establish categoricity of the natural numbers in its open-ended schematic formulation in a simpler way that is also conservative over PRA. For an informal discussion of the categoricity of initial segments of the cumulative hierarchy of sets in the spirit of open-ended axiom systems, see Martin (2001), sec. 3.

even when that is enlarged by the successive adjunction of consistency statements transfinitely iterated over such well-orderings. What confidence are we to have in the resulting purely arithmetical theorems? There is hardly any reason to doubt the consistency of PA^2 itself, even though by Gödel's second incompleteness theorem, we cannot prove it by means that can be reduced to PA^2. Indeed, the ideal world picture of $(N, S(N), \in)$ that we have been countenancing would surely lead us to say more, since in it the natural numbers are taken in their standard conception. On this account, any arithmetical statement that we can prove in PA^2 ought simply to be accepted as true. But given that the assumption of $S(N)$ as a definite totality is a purely hypothetical and philosophically problematic one, the best we can rightly say is that *in that picture*, everything proved of the natural numbers is true.

Incidentally, all of this and more comes into question when we move one type level up to the structure $(N, S(N), S(S(N)), \in_1, \in_2)$ in which Cantor's continuum hypothesis may be formulated. A more extensive discussion of the conception of that structure and the question of its definiteness in connection with the continuum problem is given in Feferman (2011). We shall also see below how taking N and $S(N)$ to be definite but $S(S(N))$ to be open-ended can be treated in suitable formal systems.

Where and why classical first-order logic? Logic, as I affirmed at the outset, is supposed to provide us with all those forms of reasoning that lead invariably from truths to truths, i.e. it is given by an essential combination of inferential and semantical notions. But from the point of view of conceptual structuralism, the classical notion of truth in a structure need not be applicable unless we are dealing with a conception (such as that of the structure of natural numbers) for which the basic domains are definite totalities and the basic notions are definite operations, predicates and relations. It is clear that at least the classical first-order predicate calculus should be admitted both on semantical and inferential grounds, since we have Gödel's completeness theorem to provide us with a complete inferential system. *But why not more*? For example, model-theorists have introduced generalized quantifiers such as the cardinality quantifiers $(Q_\kappa x)P(x)$ expressing that there are at least κ individuals x satisfying the property P, where κ is any infinite cardinal; one could certainly consider adjoining those to the first-order formalism. A much more general class of quantifiers defined by set-theoretical means was introduced by Lindström (1966); each of those can be used to extend first-order logic with a model-theoretic semantics for arbitrary first-order domains. But for which such extensions do we have a completeness theorem like that of Gödel's for first-order logic? It is well known that no such theorem is possible for the quantifier $(Q_\omega x)P(x)$ which expresses that there are infinitely many x such that $P(x)$. For, using that quantifier and thence its dual ("there are just finitely many x such that $P(x)$") we can characterize the structure of natural numbers up to isomorphism, so all the truths of that structure are valid sentences in the logic. But the set of such truths is not effectively enumerable, indeed far from it, so it is not given by an effectively specified formal system of reasoning.

Surprisingly, Keisler (1970) obtained a completeness theorem for the quantifier $(Q_\kappa x)P(x)$ when κ is any uncountable cardinal; as it happens, that has the same set of valid formulas as for the case that κ is the first uncountable cardinal. In view of the leap over the case $\kappa = \omega$, one may suspect that the requirement that the set of valid formulas be given by some effective set of axioms and rules of inference is not sufficient to express completeness in the usual intended sense. We need to say something more about how such axioms and rules of inference ought specifically to be complete for a given quantifier. The key is given by Gentzen's (1935) system of natural deduction NK (or sequent calculus LK) where each connective and quantifier in the classical first-order predicate calculus is specified by Introduction and Elimination rules for that operation only. Moreover, for each pair of such rules, any two connectives or quantifiers satisfying them are equivalent, i.e. they implicitly determine the operator in question. So a strengthened condition on a proposed addition by a generalized quantifier Q to our first-order language is that it be given by axioms and rules of inference for which there is at most one operator satisfying them. That was the proposal of Zucker (1978) in which he gave a theorem to the effect that any such quantifier is definable in the first-order predicate calculus. In particular, that would apply to the Lindström quantifiers. However, there were some defects in Zucker's statement of his theorem and its proof; I have given a corrected version of both in Feferman (t.a.). To summarize: we have fully satisfactory semantic and inferential criteria for a logic to deal with structures whose domains are first-order and that are completely definite in the sense described above, and these limit us to the standard first-order classical logic.

Let us turn now to conceptions of structures with second-order or higher order domains, such as $(N, S(N), \in, \ldots)$ where the ellipsis indicates that this augments an arithmetical structure on N such as $(N, 0, Sc, <, +, \times)$. Again, if $S(N)$ is considered as a definite totality, the classical notion of truth is applicable and the semantics of second-order logic must be accepted. But as is well known there is no complete inferential system that accompanies that, since again the arithmetical structure is categorically axiomatized in this semantics and in consequence the set of its truths is not effectively enumerable. In any case, as I have argued above, $S(N)$ ought not to be considered as a definite totality; to claim otherwise, is to accept the problematic realist ontology of set theory. As Quine famously put it, second-order logic is "set theory in sheep's clothing." Boolos (1975, 1984) tried to get around this via a reduction of second-order logic to a "nominalistic" system of plural quantification. This was incisively critiqued by Resnik in his article "Second-order logic still wild": "Boolos is involved in a circle: he uses second-order quantification to explain English plural quantification and uses this, in turn, to explain second-order quantification." (Resnik 1988, p. 83).

Though the Lindström quantifiers are restricted to apply to first-order structures and thus bind only individual variables they may well be defined using higher order notions in an essential way, in particular those needed for the cardinality quantifiers. Another example where the syntax is first-order on the face of it but the semantics is decidedly second-order is IF ("Independence Friendly") logic, due to Hintikka (1996). This uses formulas in whose prenex form the existentially quantified individual variables are declared to depend on a subset of the universally quantified individual variables that precede it in the prefix list. Explanation of the semantics of this requires the use of quantified function variables; over any given first-order structure $(D, \ldots)$ those variables are interpreted to range over functions of various arities with arguments and values in D. Indeed, Väänänen (2001) p. 519 has proved that the general question of validity of IF sentences is recursively isomorphic to that for validity in full second-order logic. Thus, as with the

Lindström quantifiers, the formal syntax can be deceptive. See Feferman (2006) for an extended critique of IF logic.

Where and why intuitionistic first-order logic? Now let us turn to the question which logic is appropriate to structural conceptions that are taken to lack some aspect of definiteness. Offhand, one might expect the answer in that case to be intuitionistic logic, but the matter is more delicate. The problem is that there is not one clear-cut semantics for it; among others that have been considered, one has the so-called BHK interpretation, Kripke semantics, topological semantics, sheaf models, etc., etc. Of these, the first is the most principled one with respect to the basic ideas of constructivity; it is that that leads one directly to intuitionistic logic but it does not determine it via a precise completeness result. By contrast, as we shall see, not only does Kripke semantics take care of the latter but it relates more closely to the question of dealing with conceptions of structures involving possibly indefinite notions and domains. For the details concerning both of these I refer to Troelstra and van Dalen (1988), a comprehensive exposition of constructivism in mathematics that includes treatments of the great variety of semantics and proof theory that have been developed for intuitionistic systems.

The BHK (Brouwer-Heyting-Kolmogoroff) constructive explanation of the connectives and quantifiers is described in Troelstra and van Dalen (1988), p. 9. It uses the informal notions of construction and constructive proof; for each form of compound statement C necessary and sufficient conditions are provided on what it is for a construction to be a proof of C, in terms of proofs of its immediate sub-statements. Namely, a proof of $A \wedge B$ is a proof of A and a proof of B; a proof of $A \vee B$ is a proof of A or a proof of B; a proof of A $\rightarrow$ B is a construction that transforms any proof of A into a proof of B; and a proof of $\neg A$ is a construction that transforms any proof of A into a proof of a contradiction $\bot$, i.e. is a proof of $A \rightarrow \bot$. In the case of the quantifiers, where the variables range over a given domain D, a proof of $(\forall x)A(x)$ is a construction that transforms any d in D into a proof of $A(d)$; finally, a proof of $(\exists x)A(x)$ is given by a d in D and a proof of $A(d)$. (D must be a constructively meaningful domain, so that it makes sense to exhibit each individual element of D and for constructions to be applicable to elements of D.)

A statement A of the first-order predicate calculus is constructively valid according to the BHK interpretation if there is a proof of A, independently of the interpretation of the domain D and the interpretation of the predicate symbols of A in D. The axioms of intuitionistic logic in any of its usual formulations are readily recognized to be constructively valid and the rules of inference preserve constructive validity. But since there are no precise notions of proof and construction at work here, we cannot state a completeness result for the BHK interpretation. Instead, the literature uses "weak counterexamples" to show why it is plausible on that account that a given classically valid form of statement is not constructively valid. Thus, for example, to show that $A \vee \neg A$ is not constructively valid as a general principle one argues that otherwise one would have a general method for obtaining for any given statement A, either a proof of A or a proof that turns any hypothetical proof of A into a contradiction. But if we had such a universal method, we could apply it to any particular statement A that has not yet been settled, such as the twin prime conjecture, to determine its truth or falsity. Similarly, the method of weak counterexamples is used informally to argue against the constructive validity of many other such schemes, for example $\neg\neg A \rightarrow A$, though the converse is recognized to be valid.[5]

Let us turn now to Kripke semantics for the language of first-order predicate logic (Troelstra and van Dalen 1988, Ch. 2.5-2.6). A Kripke model is a quadruple $(K, \leq, D, v)$, where (i) $(K, \leq)$ is a non-empty partially ordered set, (ii) D is a function that assigns to each k in K a non-empty set $D(k)$ such that if $k \leq k'$ then $D(k) \subseteq D(k')$, and (iii) v is a function into $\{0, 1\}$ at each k in K, each n-ary relation symbol R in the language and n-ary sequence of elements of $D(k)$, such that if $k \leq k'$ and $d_1,\ldots,d_n \in D(k)$ and $v(k, R(d_1,\ldots,d_n)) = 1$ then $v(k', R(d_1,\ldots,d_n)) = 1$. One motivating idea for this is that the elements of K represent stages of knowledge, and that $k \leq k'$ holds if everything known in

[5] Various methods of realizability, initially introduced by Kleene in 1945, can be used to give precise independence results for such schemes, but are still not complete for intuitionistic logic. Cf. Troelstra and van Dalen (1988), Ch. 4.4.

stage k is known in stage k'. Also, $v(k, R(d_1,\ldots,d_n)) = 1$ means that $R(d_1,\ldots,d_n)$ has been recognized to be true at stage k; once recognized, it stays true. The domain $D(k)$ is the part of a potential domain that has been surveyed by stage k; the domains may increase indefinitely as k increases or may well bifurcate in a branching investigation so that one cannot speak of a "final" domain in that case.

The valuation function v is extended to a function $v(k, A(d_1,\ldots,d_n))$ into $\{0, 1\}$ for each formula $A(x_1,\ldots,x_n)$ with n free variables and assignment $(d_1,\ldots,d_n)$ to its variables in $D(k)$; this is done in such a way that if $k \leq k'$ and $d_1,\ldots,d_n \in D(k)$ and $v(k, A(d_1,\ldots,d_n)) = 1$ then $v(k', A(d_1,\ldots,d_n)) = 1$. The clauses for conjunction, disjunction and existential quantification are just like those for ordinary satisfaction at k in $D(k)$. The other clauses are (ignoring parameters): $v(k, A \to B) = 1$ iff for all $k' \geq k$, $v(k', A) = 1$ implies $v(k', B) = 1$; $v(k, \bot) = 0$; and $v(k, \forall x\, A(x)) = 1$ iff for all $k' \geq k$ and d in $D(k)$, $v(k', A(d)) = 1$. As above, we identify $\neg A$ with $A \to \bot$; thus $v(k, \neg A) = 1$ iff for all $k' \geq k$, $v(k', A) = 0$. We say that k *forces* A if $v(k, A) = 1$; i.e. A is recognized to be true at stage k no matter what may turn out to be known at later stages. A formula $A(x_1,\ldots,x_n)$ is said to be valid in a model $(K, \leq, D, v)$ if for every k in K and assignment $(d_1,\ldots,d_n)$ to its free variables in $D(k)$, $v(k, A(d_1,\ldots,d_n)) = 1$. Then the completeness theorem for this semantics is that a formula A is valid in all Kripke models iff it is provable in the first-order intuitionistic predicate calculus. We shall see in the next section how Kripke models can be generalized to take into account differences as to definiteness of basic relations and domains.

Satisfying as this completeness theorem may be, there remains the question whether one might not add connectives or quantifiers to those of intuitionistic logic while retaining some form of its semantics. Though intuitionistic logic is part of classical logic, the semantical and inferential criterion above for classical logic doesn't apply because of the differences in the semantical notions. But just as for the classical case, on the inferential side each of the connectives and quantifiers of the intuitionistic first-order predicate calculus is uniquely identified via Introduction and Elimination rules in Gentzen's natural deduction system NJ. Even more, Gentzen first formulated the idea that the *meaning* of each of the above operations is given by its characteristic inferences. Actually, Gentzen claimed more: he wrote that "the [Introduction rules] represent, as it were, the 'definitions' of the symbols concerned" (Gentzen 1969, p. 80). Prawitz supported this by means of his Inversion Principle (Prawitz 1965, p. 33): namely, it follows from the normalization theorem for NJ that each Elimination rule for a given operation can be recovered from the appropriate one of its Introduction rules when that is the last step in a normal derivation. Without subscribing at all to this proposed reduction of semantics to inferential roles, we may ask whether any further operators may be added via suitable Introduction rules. The answer to that in the negative was provided by the work of Zucker and Tragesser (1978) in terms of the adequacy of what they call *inferential logic*, i.e. of the logic of operators that can simply be marked out by Introduction rules. As they show, every such operator is defined in terms of the connectives and quantifiers of the intuitionistic first-order predicate calculus. To be more precise, this is shown for Introduction rules in the usual sense in the case of possible propositional operators, while in the general case of possible operators on propositions and predicates—now in accord with the BHK interpretation—"proof" parameters and constructions on them are incorporated in the Introduction rules, but those are eventually suppressed.[6]

Semi-intuitionism: the logic of partially open-ended structures. An immediate generalization of Kripke structures is to allow many-sorted domains, possibly infinite in number. Let I be a collection of sorts. Then the definition of Kripke structure is modified to have each of K, $\leq$, and D indexed by I, and the valuation function modified to accord with the different sorts. Thus we deal with n-tuples $k = (k_1,\ldots,k_n)$ where k_m is of specified sort i_m; the $\leq$ relation then holds between such n-tuples if it holds term-wise. Of course the basic predicates come with specified arities to show what sorts of objects they relate, and the variables in the first-order language over these predicates are always

[6] Incidentally, as Zucker and Tragesser show (p. 506), not every propositional operator given by simple Introduction rules has an associated Elimination rule; a counterexample is provided by $(A \to B) \vee C$.

of a specified sort. Then the definition of the valuation function on arbitrary formulas for a many-sorted structure (K, I, $\leq$, D, v) proceeds in the same way as above. Now an n-ary relation R may be considered to be *definite* if $v(\mathrm{k}, R(d_1,\ldots,d_n)) = v(\mathrm{k}', R(d_1,\ldots,d_n))$ whenever $k \leq k'$. A domain D_i is *definite* if $D_i(k) = D_i(k')$ for all k and k' in K_i, otherwise *indefinite* or *open-ended.* While the formulas valid in the structure obey intuitionistic logic in general, one may apply classical logic systematically to formulas involving definite relations as long as the quantified variables involved range only over definite domains.

This is illustrated by reasoning about the ordinary two-sorted structure (N, $S(N)$, $\in$, …) where (N, …) is conceived of as definite with definite relations, while $S(N)$ is conceived of as open-ended. To treat this as a two-sorted Kripke structure, take $I = \{0, 1\}$ where N is of sort 0 and $S(N)$ is of sort 1. We may as well take K_0 to consist of a single element, while K_1 could be indexed by all collections k of subsets of N, ordered by inclusion. Now the membership relation is definite because sets are taken to be definite objects, i.e. if X is in both the collections k and k' then $n \in X$ holds in the same way whether evaluated in k or in k'. So classical logic applies to all formulas A that contain no bound set variables, though they may contain free set variables, i.e. A is what is usually called a *predicative formula*. But when dealing with formulas in general, only intuitionistic logic is justified on this picture. This leads us to the consideration of *semi-intuitionistic* (or *semi-constructive*) *theories* in general, i.e. theories in which the basic underlying logic is intuitionistic, but classical logic is taken to apply to a class of formulas distinguished by containing definite predicates and quantified variables ranging over definite domains. A number of such theories have been treated in the paper Feferman (2010), corresponding to different structural notions in which certain domains are taken to be definite and others indefinite. They fall into three basic groups: (i) predicative theories, (ii) theories of countable (tree) ordinals, and (iii) theories of sets. The general pattern is that in each case one has a semi-intuitionistic version of a corresponding classical system, and they are shown to be proof-theoretically equivalent and to coincide on the classical part. Moreover, the same holds when the semi-intuitionistic system is augmented by various principles such as the Axiom of Choice (AC) that would make the corresponding classical system much stronger. It is not possible here to explain the results in adequate detail, so only some of the ideas behind the formulations of the systems involved is sketched. The reader who prefers to avoid even the technicalities that remain can easily skim (or even skip) the rest of this section.

(i) *Semi-intuitionistic predicative theories.* Here the language of arithmetic is extended by variables for function(al)s in all finite types; following Gödel (1958, 1972) in his so-called *Dialectica* interpretation, we also add primitive recursive functionals in all finite types. In many-sorted intuitionistic logic, the system obtained is denoted HA^{ω}. In the process of obtaining reduction to a quantifier-free system, Gödel showed that this system is of the same strength as Peano Arithmetic, PA; in fact the same holds for HA^{ω} + AC. Now the latter is turned into a semi-intuitionistic system by adding the Law of Excluded Middle for all arithmetical formulas. For the proof-theoretical work on that, it proves to be more convenient to add the *least-number operator* μ and an axiom (μ) that says that when the operator is applied to a function f: $N \rightarrow N$ for which there exists an n with $f(n) = 0$, it yields the least such n. Under this axiom, all arithmetical formulas become equivalent to quantifier-free (QF) formulas, for which the LEM then holds. Thus one is led to consider HA^{ω} + AC + (μ), which turns out to be proof-theoretically equivalent to PA^{ω} + QF-AC + (μ), and both are equivalent to ramified analysis through all ordinals less than Cantor's ordinal ε_0. If one adds the Bar Rule for arithmetical orderings in both the semi-intuitionistic and the classical systems, we obtain systems of proof-theoretical strength full predicative analysis, i.e. ramified analysis up to the least impredicative ordinal Γ_0. (The Bar Rule on an ordering rule allows us to infer transfinite induction w.r.t. arbitrary formulas from well-foundedness of the ordering.) On the other hand, if in the basic system we restrict the primitive recursive functionals to those with values in N and restrict induction to QF formulas, we obtain a semi-intuitionistic system Res-HA^{ω} + AC + (μ) that turns out to be of exactly PA in strength.

(ii) *Semi-intuitionistic theories of countable tree ordinals.* By *countable tree ordinals* one means the members of the open-ended collection Ω of countably branching well-founded trees. Add a sort for the members of Ω to the preceding systems; extend the higher type variables accordingly; add the operator of supremum that joins a sequence of trees $f: N \to \Omega$ into a single tree $sup(f)$ in Ω; add the inverse operator that takes each $sup(f)$ in Ω and n in N and produces $f(n)$; and, finally, add operators for transfinite recursion on Ω. The resulting system is denoted SO^{ω} in intuitionistic logic and CO^{ω} in classical logic; then $SO^{\omega} + (\mu)$ is a semi-intuitionistic system intermediate between these two. The main result in this case is that the following are of the same proof-theoretical strength: SO^{ω} + AC + (μ), CO^{ω} + QF-AC + (μ), and ID_1, the theory of arbitrary arithmetical inductive definitions. It is known that the latter has the same proof-theoretical strength in intuitionistic logic as in classical logic.

(iii) *Semi-intuitionistic theories of sets.* We turn finally to the picture of the cumulative hierarchy structure, the standard classical view of which leads us to the system ZFC, i.e. ZF + AC. However, if we identify definite totalities with sets then by Russell's paradox, the "universe V of all sets" must be considered to be an open-ended indefinite totality if we are to avoid contradiction. But in the Separation Axiom scheme for ZF, $\forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge A(x)]$, one allows the formula A to contain bound variables that range without restriction over V, and hence in general do not represent definite properties; the same criticism applies to the formulas $A(x, y)$ in the Replacement Axiom scheme. By a Δ_0 formula is meant one in which all quantified variables are restricted, i.e. take the form $\forall y(y \in x \to \ldots)$ or $\exists y(y \in x \wedge \ldots)$, written respectively $(\forall y \in x)(\ldots)$ and $(\exists y \in x)(\ldots)$. The system KP of Kripke-Platek set theory in classical logic has, like ZF, the axioms of extensionality, ordered pair, union, infinity, and the scheme of transfinite induction on the membership relation. In place of the Separation Axiom scheme it takes Δ_0-Separation, i.e. the Separation Axiom scheme restricted to Δ_0 formulas. And in place of the Replacement Axiom scheme, it takes what is called Δ_0-Collection, i.e. the scheme that for each Δ_0 formula A, $(\forall x \in a)\exists y A(x, y) \to \exists b(\forall x \in a)(\exists y \in b)A(x, y)$. This implies the Replacement Axiom scheme for Δ_0 formulas. It is known that the system KP is of the same strength as ID_1.

The system IKP is taken to be the same as KP but restricted to intuitionistic logic. It turns out that we can strengthen it considerably by adding a bounded form AC_S of the Axiom of Choice, namely $(\forall x \in a)\exists y A(x, y) \rightarrow \exists f[Fun(f) \wedge (\forall x \in a)A(x, f(x))]$, where $Fun(f)$ expresses that the set f is a function in the set-theoretical sense, and where now A is an *arbitrary* formula of the language of set-theory. Under the assumption AC_S we can infer Collection for arbitrary formulas and hence Replacement for arbitrary formulas. Finally, since sets are considered to be definite totalities, we obtain a semi-intuitionistic system from IKP by adjoining the law of excluded middle for Δ_0 formulas. The main result of Feferman (2010) is that the semi-intuitionistic system IKP + AC_S + Δ_0-LEM is of the same proof-theoretical strength as KP and hence of ID_1 in its classical and intuitionistic forms. Moreover, if we add the Power Set Axiom (Pow) we obtain a system that is of strength between that of KP + Pow and that of KP + Pow + (V = L).[7,8]

It is natural in the context of semi-intuitionistic theories T to say that a sentence A in the language of T is *definite* (relative to T) if T proves LEM for A, i.e. $A \vee \neg A$. A question in set theory that has caused considerable discussion in recent years is whether Cantor's Continuum Hypothesis CH is a definite mathematical problem. One formulation of it is that every subset of $S(N)$ is either countable or in 1-1 correspondence with $S(N)$. Of course, that is definite in the theory IKP + Pow + Δ_0-LEM , because quantification over subsets of $S(N)$ is bounded once we have existence of $S(S(N))$ [i.e., $S(S(\omega))$] by the Power Set Axiom. That suggests—as I did in Feferman (2012)—considering the weaker system T = IKP + Pow(N) +AC_S + Δ_0-LEM, where Pow(N) simply asserts the existence

[7] There is a considerable literature on semi-intuitionistic theories of sets including the power set axiom going back to the early 1970s. See Feferman (2010) sec. 7.2 for references to the relevant work of Poszgay, Tharp, Friedman, and Wolf.
[8] Mathias (2001) proved that KP + Pow + (V = L) proves the consistency of KP + Pow, so the usual argument for the relative consistency of (V = L) doesn't work.

of $S(N)$ as a set. I conjectured there that CH is not definite relative to that system.[9] Of course, that would not show that CH is not a definite mathematical problem, but it might be considered as an interesting bit of evidence in support of that.

Conceptual structuralism and mathematical practice. One criterion for a philosophy of mathematics that is often heard is that it should accord with mathematical practice. It's very hard to know just what that means since there are so many dimensions along which practice can be viewed. One particular interpretation of the criterion is that philosophers have no business telling mathematicians what does or doesn't exist. Famously, David Lewis wrote:

> I'm moved to laughter at the thought of how *presumptuous* it would be to reject mathematics for philosophical reasons. How would *you* like the job of telling the mathematicians that they must change their ways, and abjure countless errors, now that *philosophy* has discovered that there are no classes? (Lewis 1991, p. 59)[10]

But this is a caricature of what philosophy is after; philosophers take for granted that mathematicians have settled problematic individual questions of existence like zero, negative numbers, imaginary numbers, infinitesimals, points at infinity, probability of subsets of [0, 1], etc., etc., using purely mathematical criteria in the course of the development of their subject. The existence of some of these has been established by reduction to objects whose existence is unquestioned, some by qualified acceptance, and some not at all. But what the philosopher is concerned with is, rather, to explain in what *metaphysical* sense, if any, mathematical objects exist, in a way that cannot even be discussed within ordinary mathematical parlance. Lewis could equally well have laughed at the idea that some general principles accepted in the mathematical mainstream such as the Law of Excluded Middle or the Axiom of Choice would be dismissed as false (or

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[9] Michael Rathjen has recently announced a proof of this conjecture (private communication).

[10] Curiously, this quote is from Lewis' book, *Parts of Classes*, which offers a revisionary theory of classes that differs from the usual mathematical conception of such.

unjustified) for philosophical reasons. But again, the use of truth in ordinary mathematical parlance is deflationary and the reasons for accepting such and such principles as true has either been made without question or for mathematical reasons in the course of the development of the subject. The philosopher, by contrast, is concerned to explain in what sense the notion of truth is applicable to mathematical statements, in a way that cannot be considered in ordinary mathematical parlance. Whether the mathematician should pay attention to either of these aims of the philosopher is another matter.

Conceptual structuralism addresses the question of existence and truth in mathematics in a way that accords with both the historical development of the subject and each individual's intellectual development. It crucially identifies mathematical concepts as being embedded in a social matrix that has given rise, among other things, to social institutions and games; like them, mathematics allows substantial intersubjective agreement, and like them, its concepts are understood without assuming reification.[11] What makes mathematics unique compared to institutions and games is its endless fecundity and remarkable elaboration of some basic numerical and geometrical structural conceptions. To begin with, mathematical objects exist only as conceived to be elements of such basic structures. The direct apprehension of these leads one to speak of truth in a structure in a way that may be accepted uncritically when the structure is such as the integers but *may* be put into question when the conception of the structure is less definite as in the case of the geometrical plane or the continuum, and *should* be put into question when it comes to the universe of sets. One criticism of conceptual structuralism that has been made is that it's not clear/definite what mathematical concepts are clear/definite, and making that a feature of the philosophy brings essentially subjective elements into play.[12] Actually, conceptual structuralism by itself, as presented in the theses 1-10, takes no specific position in that respect and recognizes that different judgments (such as mine) may be made. Once such are considered, however, logic has much to tell us in its role as

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[11] For an interesting social institutional account of mathematics see Cole (2013); this differs from conceptual structuralism in some essential respects while agreeing with it in others.

[12] In particular, this criticism has been voiced by Peter Koellner in his comments on Feferman (2011); cf. http://logic.harvard.edu/EFI_Feferman_comments.pdf.

an intermediary between philosophy and mathematics. As shown in the preceding section, one can obtain definitive results about formal models of different standpoints as to what is definite and what is not. Moreover, the results can be summarized as telling us that to a significant extent, the unlimited (*de facto*) application of classical logic in mainstream mathematics—i.e., the logic of definite concepts and totalities—may be justified on the basis of a more refined mixed logic that is sensitive to distinctions that one might adopt between what is definite and what is not.[13] In other words, once more they show that, at least to that extent, you can have your cake and eat it too.

There are other dimensions of mathematical practice that reward metamathematical study motivated by the philosophy of conceptual structuralism. One, in particular, that I have emphasized over the years is the open-ended nature of certain principles such as that of induction for the integers and comprehension for sets. This accords with the fact that in the development of mathematics what concepts are recognized to be definite evolve with time. Thus one cannot fix in advance all applications of these open-ended schematic principles by restriction to those instances definable in one or another formal language, as is currently done in the study of formal systems. This leads instead to the consideration of logical models of practice from a novel point of view that yet is susceptible to metamathematical study. One such is via the notion of the *unfolding of open-ended schematic axiom systems*, that is used to tell us everything that ought to be accepted if one has accepted given notions and principles. Thus far, definitive results about the unfolding notion have been obtained by Feferman and Strahm (2000, 2010) for schematic systems of non-finitist and finitist arithmetic, resp., and by Buchholtz (2013) for arithmetical inductive definitions. As initiated in Feferman (1996), I am optimistic that it can be used to elaborate Gödel's program for new axioms in set theory and in particular to draw a sharper line between which such axioms ought to be accepted on intrinsic grounds and those to be argued for on extrinsic grounds.

[13] These kinds of logical results can also be used to throw substantive light on philosophical discussions as to the problem of quantification over everything (or over all ordinals, or all sets) such as are found in Rayo and Uzquiano (2006).

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