Introduction

[To *Foundations of Explicit Mathematics* (in progress), by Solomon Feferman, Gerhard Jäger, and Thomas Strahm, with the assistance of Ulrik Buchholtz]

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What is Explicit Mathematics?

Explicit Mathematics is a flexible unified framework for the systematic logical study of those parts of higher mathematics in which proofs of existence guarantee the computability or definability by specified means of what is demonstrated to exist. It would seem that such parts of mathematics must be relatively restricted, given the ubiquity of existence proofs throughout modern mathematics for which no method is known, either in practice or in principle, to produce the objects asserted to exist. Indeed, the main parts of mathematics covered by the Explicit Mathematics framework are referred to as *constructive*, *predicative*, and *descriptive* in the senses that will be described below, and each was originally pursued on philosophical grounds that mathematicians for the most part have not found persuasive and too confining for the purposes of practice. What is not generally known and will be revealed in the present work through the logical analysis provided by our framework is that in gaining the uniform explicitness of solutions one does not pay a great price in terms of both the workability and mathematical reach of these approaches, despite their philosophical and methodological restrictions. In particular, a weak predicative system already serves to account for all scientifically applicable mathematics in current use. Though the constructive, predicative and descriptive approaches are the main parts of mathematics with which this book is concerned, Explicit Mathematics has proved to be adaptable to a variety of other contexts ranging from theories of feasible computation and finitist mathematics to large cardinals in set theory, as will be described in the final part of this book.

The Origins of Explicit Mathematics

I was led to the initial development of Explicit Mathematics in the mid 1970s when trying to understand what Errett Bishop had accomplished in his groundbreaking work, *Foundations of Constructive Analysis* (1967). His constructive redevelopment of analysis went much farther in the subject than anything that had been previously accomplished in the school of Brouwerian intuitionism and its variants. The culprit in non-constructive existence proofs had been identified by Brouwer to lie in the general application of the method of proof by contradiction: to establish $\exists x A(x)$, assume its negation $\neg\exists x A(x)$ and show that that leads to a falsehood. That method in turn depends in an essential way on the assumption of the Law of the Excluded Middle (LEM). The general use of LEM, except for effectively decidable properties, was thus excluded from intuitionistic logic. It was later shown by the so-called "realizability" interpretations introduced by Kleene in 1945 that in a suitable sense, intuitionistic number theory is compatible with the assumption that every function proved to exist is recursive, contrary to what holds in classical number theory. Though Bishop agreed with the Brouwerians that one should restrict oneself to reasoning in intuitionistic logic, I came to the conclusion that that was not the real reason why one could give a systematic recursive interpretation to his results. Rather, its success in that respect depends essentially on two features, one general and the other more specific. The general point is that all of Bishop's basic notions are considered without assumption of extensionality, and in that sense are *intensional*, although in an abstract sense. (It is that which the 'Explicit', in 'Explicit Mathematics', is intended to suggest.) In particular, operations can be interpreted directly as computational programs, or indices of partial recursive functions. But they can also be considered extensionally, thus making the basic notions a part of classical mathematics. The second, more specific, feature of Bishop's methodology that, in my view, accounted for the success of his approach, was the way he modified classical notions to incorporate certain "witnessing data" that is implicitly carried along in proofs. Together, his notions and results may be considered to be a refinement of classical mathematics that at the same time admits of a constructive interpretation in recursive form. Since Bishop's redevelopment of analysis is simply a part of classical mathematics, that is another way in which it diverges significantly from Brouwerian intuitionism. Brouwer treated real numbers as "choice sequences" of rational numbers, of which one would only have a finite amount of information at any time, and functions of real numbers would thus be recast in terms of functions of choice sequences. From this, Brouwer was led to the theorem that *every* function on a closed interval is continuous, patently contradicting classical analysis. Bishop's approach, by contrast, admits dealing with discontinuous (partial) functions on the real numbers in his theory of measure.

From this reading of Bishop style constructive analysis, I was led to introduce an axiomatic system T_0 based on classical logic in which all his work could be directly formalized. As in Bishop's work, the ontology of T_0 was taken to be that of a universe of objects including (i) the *natural numbers*, (ii) *operations* (in general *partial*) and (iii) *classes*[1] (a natural extension of Bishop's sets); moreover, operations and classes are to be understood as given intensionally. Operations of pairing and projection are taken as basic, and operations can be applied to any objects in the universe, including operations and classes. For example, we have an operation f which takes any pair X, Y of classes to produce their cartesian product, $X \times Y$ and another operation g which takes X, Y to the cartesian power Y^X, also written $X \rightarrow Y$. The formation of such classes is governed by an *Elementary Comprehension Axiom* scheme (ECA) which tells which properties determine classes in a uniform way from given classes; these are given by formulas in which classes may be used as parameters to the right of the membership relation and in which we do not quantify over classes. But to form general products we need further notions and an additional axiom. Given a class I, by an I-termed sequence of classes is meant an operation f with domain I such that for each $i \in I$ the value of f(i) is a class X_i; one wishes to use this to define $\Pi X_i[i \in I]$. It turns out that in combination with ECA a more basic operation is that of forming the disjoint sum or join $\sum X_i[i \in I]$, but an additional axiom called *Join* is needed to assure its existence. Using lower case letters a, b, c,…, x, y, z for objects in general— among which operations f, g, h,…—and upper case letters A, B, C, …, X, Y, Z for classes, the ontology just described requires the

[1] In the original publication on Explicit Mathematics, Feferman (1975), I used the word *classifications* for what I simply referred to as *classes* in the follow-up publication, Feferman (1979).

curious looking axiom $\forall X \exists x(X = x)$. Though it may seem peculiar at first sight, it is reasonable when one considers that all the objects with which one deals may be conceived of as given intensionally, i.e. by means of presentations that can be coded by individuals.

In the approach to the formalization of Explicit Mathematics due to Jäger (1988), it turned out to be more convenient to treat classes extensionally but with many possible representations within the universe V of individuals; that is the approach that will be followed in this book.[2] Membership has its usual meaning, but a new basic relation is needed, namely that an object x *names* or *represents* the class X, written $\mathfrak{R}(x, X)$. In these terms, for example, one has operations f and g such that whenever $\mathfrak{R}(x, X)$ and $\mathfrak{R}(y, Y)$ hold then $\mathfrak{R}(f(x, y), X \times Y)$ and $\mathfrak{R}(g(x, y), X \to Y)$ hold.

As mentioned above, operations are in general partial as in ordinary mathematics. A basic relation for these is that of *application*, App(x, y, z), which expresses that the operation x applied to y is defined and has value z; this is also written $xy \simeq z$ and alternatively as $x(y) \simeq z$. Underlying all systems of Explicit Mathematics is a simple set of axioms for the applicative (operational) structure that is given by the *partial combinatory calculus*, from which the *partial λ-calculus* is directly derived. It is convenient here to allow the unrestricted formation of terms by closure under the application operation xy, and to take a new basic relation $t\!\downarrow$ to express that the term t is defined. One must then make some simple modifications to the usual predicate calculus via what is called *The Logic of Partial Terms* (LPT), due to Beeson (1985). For example, in place of the usual axioms $\forall x A(x) \to A(t)$ one takes $\forall A(x) \wedge t\!\downarrow \to A(t)$. In these terms, for any types X and Y, $X \to Y$ consists of all z such that for all $x \in X$, $z(x)\!\downarrow$ and $z(x) \in Y$.

Bishop-style constructive analysis within the framework of Explicit Mathematics

More features of T_0 and its reorganization in this book will be described below. Meanwhile, let us return to the formalization of Bishop-style constructive analysis within

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[2] There is a difference in terminology, though. Jäger used 'types' for our classes.

it that was one of the initial motivations for its . Starting with the natural numbers N, one can form the integers Z and rationals Q in the usual way via their representatives formed by pairing; write N^+ for the positive integers. Then Bishop defines a real number to be an N^+-termed sequence x whose values x_n (i.e., x(n)) all lie in Q and for which one has $|x_n - x_m| \leq 1/n + 1/m$ for all n, m; thus each x_n is within 1/n of the "limit" of the sequence.[3] R is defined to be the set of all real numbers; for x, y in R, one defines $x =_R y$ to hold just in case $|x_n - y_n| \leq 2/n$ for all n. A real-valued *function* of real numbers is an operation f: R→ R such that whenever $x =_R y$ then $f(x) =_R f(y)$. In classical analysis it is shown that every continuous function on a closed interval [a, b] is uniformly continuous, i.e. for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever x, y lie in [a, b] and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. For constructive purposes, Bishop instead defines a continuous function on [a, b] to be a pair (f, d) where f is a function from [a, b] to R and d: $R^+ \rightarrow R^+$ is such that whenever $\varepsilon > 0$ and $|x - y| < d(\varepsilon)$ then $|f(x) - f(y)| < \varepsilon$. In other words a continuous function incorporates as witnessing data a uniform modulus of continuity operation that is carried along when inferring its properties. This allows us, for example, to construct the sum of two continuous functions (f, d) and (g, e) to yield a new continuous function.

This is just a bare indication of how the formalization of Bishop style constructive analysis proceeds within the framework of T_0. As will be shown in Ch. 14, for most of this work only a very limited part of T_0 is needed, namely, the partial combinatory structure together with the natural numbers, the elementary comprehension principle (ECA) and induction on N limited to classes. As will be seen, the resulting system is of the same proof-theoretical strength as Peano Arithmetic, PA. Finally, there is no need to restrict to intuitionistic logic in the development of Bishop's approach and in any case we want to have classical logic as our basic system of reasoning throughout in order to deal in a common way with constructive, predicative and descriptive mathematics.

Predicative analysis in the framework of Explicit Mathematics

[3] A common alternative in the constructive literature is to take real numbers to be pairs (x, c) of sequences x: N → Q and c: N → N, for which $|x_n - x_m| \leq 2^{-k}$ for all n, m ≥ c(k); c is then a *modulus of convergence* operation.

Bishop asserted of his approach to constructive analysis that for each theorem A of classical analysis one has a constructive version A* such that A follows from A* under the assumption of what is called the *Limited Principle of Omniscience*,

(LPO) $\forall n\,(fn \neq 0) \vee \exists n\,(fn = 0)$,

where f: $N \to N$.[4] (Actually, more is needed, as will be explained in Ch. 14.) LPO is a consequence of the Law of Excluded Middle, and even that special case is rejected by the intuitionists as well as by Bishop.[5] Intuitively, given f, we can only decide number by number whether it is equal to 0 or not, but that is insufficient to decide whether or not it holds for all natural numbers. In more philosophical language, within constructive mathematics N is regarded as a *potential totality*. By contrast, in *predicative mathematics* growing out of the ideas of Henri Poincaré and initially developed by Hermann Weyl in his fundamental work, *Das Kontinuum* (1918), N is regarded as a *completed totality.* In that approach, not only is LPO accepted, but every operation which can be defined by quantification over N is accepted too. The basic mathematical operation which guarantees this is the so-called *unbounded minimum operator* μ, which is defined for each operation f: $N \to N$, and is least such that the following condition holds:

(μ) $\exists n(f(n) = 0) \to f(\mu f) = 0$.[6]

Then of course

$$\exists n(f(n) = 0) \leftrightarrow f(\mu f) = 0 \ \text{ and } \ \forall n\,(fn \neq 0) \leftrightarrow f(\mu f) \neq 0,$$

so that every arithmetical statement is decided.

[4] Cf. Bishop and Bridges (1985), pp. 11-12.

[5] LPO is also not recursively realizable when we take f to be a free function variable in arithmetic.

[6] Alternatively, we can use the operation E_0 which is such that for each operation f that is total on N, $E_0(f)$ is 0 or 1, and $E_0(f) = 0$ if and only $\exists n(f(n) = 0)$.

An example from number theory, namely the famous Hilbert/Waring theorem, illustrates a basic difference between constructive and predicative mathematics. Lagrange proved in the 18th century that every positive integer is the sum of at most four squares; three squares do not suffice in general. Waring then made the bold conjecture that for each n there exists a k such that every integer m is the sum of at most k nth powers. Hilbert proved Waring's conjecture in 1909. Let f(n) be the least k for which the Hilbert/Waring theorem holds for all sufficiently large integers m.[7] The only n for which the exact value of f(n) has been determined are f(2) = 4 and f(4) = 16; for example, for n = 3 it is only known that $4 \leq f(3) \leq 7$. On the face of it, f is not a constructively defined function and may well be recursively incomputable. On the other hand, f is a perfectly well defined function for the predicativist.

The system T_1 of Explicit Mathematics is obtained from T_0 by adjoining the above axiom (μ). Its applicative axioms can then be interpreted in terms of what is called *recursion in the μ operator*.[8] Basic results of that theory show that the total number-theoretic functions recursive in μ are exactly the hyperarithmetical (HYP) functions, and the partial functions recursive in μ are exactly the $\prod^1{}_1$ functions. The HYP functions are also viewed as obtained by iteration of the so-called jump operation through the recursive ordinals.

The development of predicative analysis can proceed along lines similar to that of constructive analysis but with various notions simplified to be closer to their classical counterparts. But the key classical existential principle that cannot be met in full using only predicatively justified assumptions is that of the *Least Upper Bound*: this asserts that for each non-empty subset S of the real numbers that is bounded above, the least upper bound of S—in symbols, lub(S)—exists. In the case that l = lub(S) is a member of S, it appears that l is defined in terms of a totality of which it itself is one of the members. Poincaré thought that such *circular* or *impredicative* definitions lay behind the paradoxes,

[7] The letter 'G' is used for this function in the literature.

[8] Alternatively, it can be described in terms of recursion on the least admissible set containing ω.

for example in the definition of the Russell set in terms of the presumed totality of all sets, and should be systematically excluded. Or to put it in positive terms, he thought that only definitions which successively appeal only to prior defined collections of objects are to be accepted in predicative mathematics. It can be shown that the general LUB principle does not hold under that restriction. But, as Weyl demonstrated, there is an acceptable predicative weakening of the LUB principle that *is* widely applicable, namely that if $s: N \to R$ is a *sequence* of real numbers that is bounded above, then $\text{lub}\{s_n \mid n \in N\}$ exists.

The issue of the LUB principle relates to another example of a basic difference between constructive and predicative mathematics, namely the *extreme value theorem* in analysis. Let a, b be real numbers with $a \leq b$, and let f be a continuous function on the closed interval [a, b]. The classical extreme value theorem says that f attains both its maximum and its minimum on the given interval. That is, in the case of the maximum, it tells us that there exists an x such that f(x) is the maximum of the range of f on [a, b]; similarly for the minimum. The extreme value theorem was long accepted as intuitively obvious but only proved rigorously in the 19^{th} century by the methods introduced by Cauchy. However, it is shown to be non-constructive by an example due to Specker (1959) in which a continuous f on [0, 1] is defined that is computable in the sense that f(x) is recursive for each recursive x, but there is no such x at which f attains its maximum. By contrast, the extreme value theorem holds in predicative analysis.

In fact, Weyl proved that all of the results of 19^{th} century analysis can be obtained by predicative means. This has subsequently been extended to considerable parts of 20^{th} century functional analysis, including all that is currently used in scientifically applicable mathematics. Moreover, we shall see in Ch. 15 that that development can all be formalized in a relatively weak subsystem of T_1 that again turns out to be conservative over Peano Arithmetic. T_1 as a whole goes far beyond what is predicatively acceptable, even in principle. But it shares with predicativity the property that it may be considered a refinement of classical mathematics whose existential results have an explicit interpretation using the operations recursive in the unbounded minimum operator.

Descriptive set theory in the framework of Explicit Mathematics

We turn next to what may be called *descriptive* mathematics in the Explicit Mathematics framework. The classical school of *Descriptive Set Theory* (DST) emerged at the hands of some of the leading French mathematicians at the turn of the 20th century, principally Baire, Borel, and Lebesgue, in reaction to Dirichlet's idea of an arbitrary function as any many-one correspondence. They took the real numbers for granted as a completed totality and thought that one should be concerned only with functions of real numbers that are analytically definable in some sense. In particular, Baire introduced and studied in a hierarchy along the countable ordinals the smallest class that contains all the continuous functions and is closed under pointwise limits. Relatedly, Borel introduced the smallest class of sets of reals that contains all the open intervals and is closed under countable unions and complements; the Borel sets are also classified along the countable ordinals. Lebesgue introduced the concepts of measurability for sets and functions for his general theory of integration and showed that the Baire functions and Borel sets are all measurable, but that they do not exhaust the class of measurable functions and sets. In 1905 he went on to consider projections of Borel sets in the plane, and mistakenly concluded that they are again Borel. Ten years later, that error was spotted by Suslin, then a student of Lusin in Moscow. Suslin called the projections of Borel sets, *analytic sets* or A-sets and showed that they have many "good" properties, including that of Lebesgue measurability. He also showed that the Borel sets are exactly the sets that are both analytic and have an analytic complement. The analytic sets are the first level in what is called the *projective hierarchy*, obtained by closing under projections (P) and complementation (C). Thus the CA sets are also Lebesgue measurable, but efforts to extend that property to sets at higher levels in the projective hierarchy did not succeed. The reason for that was later demonstrated in 1938 by Gödel who showed that in his model L of ZFC there are non-measurable PCA sets.[9]

In the 1950s, Kleene developed effective analogues of Classical DST, using the analogy of recursive sets with the basic open sets and taking effective countable joins of sets as an

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[9] Modern DST has extended measurability and other "good" properties of sets to all levels of the projective hierarchy, under the problematic assumption of what is called *projective determinacy* or equivalently, the existence of certain very large cardinal numbers. See Martin and Steel (1989).

analogue to countable unions. Thus the hyperarithmetic (HYP) sets emerge as the analogue of the Borel sets in *Effective DST*. Then the analogue of the A-sets are those that are in the class Σ^1_1, i.e. those definable in the form $\{x \mid (\exists f)(\forall n)R(x, f|n)\}$, where $f|n$ is the number of the sequence $\langle f(0),\ldots,f(n-1)\rangle$ and $R(x, y)$ is a recursive relation. The Π^1_1 sets are then the analogue of the CA sets. Kleene's analogue of the Suslin theorem is that $HYP = \Sigma^1_1 \cap \Pi^1_1$.

All this suggests that it may be reasonable to study Classical and Effective DST under one roof in the framework of Explicit Mathematics, by formalizing DST in a suitable such system that has both classical and effective models; in fact that can be done in T_0. Not mentioned so far is that the system T_0 contains a quite general axiom IG of Inductive Generation, which we can use to generate what corresponds to the Borel sets. However in the framework of T_0, like the real numbers R, the Borel sets must be considered as certain names of classes. Then IG allows us to define a class B of names that contains the names of open intervals of reals, and is closed under an operation corresponding to countable unions. In the model of T_0 in which the applicative structure is given by the indices of partial recursive functions, the members of B turn out to be the indices of hyperarithmetic sets, while those in the model based on an applicative structure containing codes of all set-theoretic functions, the interpretation of B corresponds to the Borel sets of Classical DST. It is shown in Ch. 16 how to prove an abstract version of the Suslin-Kleene theorem in T_0 (in fact, using only IG with countable closure conditions) which thus generalizes both its classical and effective versions.

One can also consider a second approach to DST based on an extension T_2 of T_1 obtained by adding a constant for the Suslin operator E_1 that tests for any given g whether or not $(\exists f)(\forall n)g(x, f|n) = 0$. The applicative structure of T_2 can thus be taken to be the partial functions recursive in E_1. In T_2 *without* the IG axiom, we can go directly to the analytic sets, and prove a version of the Suslin-Kleene theorem there. This leads to a second kind of Effective DST, but one where the partial recursive functions are replaced by those partial recursive in the Suslin operator.

Plan of the book

In this book, the formulation of the systems T_i for i = 0, 1, 2 described above is reorganized in such a way that the work of describing models and obtaining proof-theoretical lower and upper proof-theoretical bounds is more easily handled in stages. Essentially, this is by treating pure operational (applicative) theories first and only later adding classes via the theory of classes and names.

The reader is assumed to have a general background in recursion theory and proof theory; several chapters are devoted to fixing the notation and notions that we use from these areas and stating the needed facts from the literature. In particular, **Part A, Preliminaries** (Chs. 1 and 2), sets down what we will need from recursion theory and the structure of monotone and nonmonotone inductive definitions.

Part B, Operational Theories, consists of three chapters providing the basics of applicative theories. The first of these (Ch. 3) presents LPT, the Logic of Partial Terms in quite general terms. Then the pure operational theory BON, the Basic Theory of Operations and Numbers, is introduced in Ch. 4. The universe of discourse of BON serves as a partial combinatory algebra with pairing and projections; in addition, we have a predicate N for the natural numbers, and are provided with the usual basic operations on N together with a recursor that is used to generate all primitive recursive functions. In the language of BON we meet three distinct principles of induction on N, the weakest of which is called *basic induction* (B-I_N) where one may induct on properties of N that have a characteristic function on N. The strongest form of induction considered for N, (L-I_N) is with respect to all properties definable in the language L of BON; that is also called *full induction* with respect to L. We shall also consider a special case (Op-I_N) of (L-I_N) called *operational induction*, that allows one to prove that certain operations are total on N. As we shall see, distinctions between the forms of induction on N that appear in each theory of explicit mathematics make a real difference in the associated measures of proof-theoretical strength. As will be shown in Ch. 7, it is easily seen that BON + (B-I_N) includes and is of the same proof-theoretical strength as Primitive Recursive Arithmetic, PRA, while BON + (L-I_N) includes and is of the same proof-theoretical strength as Peano Arithmetic, PA; in both cases we have conservation over the respective systems of arithmetic.

In Ch. 5 of **Part C, Reference Systems for Proof-Theoretic Bounds**, we shall remind the reader of various systems of first order and second arithmetic whose proof-theoretical strength has been precisely calibrated in the literature. Again, these strengths are quite sensitive to what forms of induction are taken in each theory. Then, as a bridge to establishing the proof-theoretical strength of our systems of Explicit Mathematics, in Ch. 6 we introduce certain theories of numbers and ordinals in which both monotone and nonmonotone inductive definitions can be directly formalized. In that case, the distinction between various forms of induction on ordinals will also play a crucial role.

In **Part D, Operational Theories Continued**, we return to applicative theories, first (Ch. 7) to relate BON and its different induction principles to subsystems of Peano Arithmetic as previously mentioned. Then in Ch. 8 we extend these systems by the type two functionals μ and E_1 described above in connection with predicative and descriptive mathematics and give their main models there, while the proof-theoretical strengths of these extensions are determined in Ch. 9. The next major step comes in **Part E, Theories of Classes and Names**, that begins in Ch. 10 with a general introduction to these theories and a uniform form of the *Elementary Comprehension Axiom*, ECA, the principal axiom of the *Elementary Theory of Classes*, EC, beyond BON. The expansion of the language also leads to new models and the formulation of new forms of induction on N that must be considered. The proof-theoretical strength of EC in combination with the various forms of induction and the functional operations is determined in Ch. 11. In particular, writing ($C\text{-}I_N$) for *class induction*, the systems EC + ($C\text{-}I_N$) and EC(μ) + ($B\text{-}I_N$) used to formalize substantial portions of constructive and predicative mathematics respectively, are both shown to be of the same strength as PA. The join and inductive generation axioms are introduced in Ch. 12 and the proof-theoretic strengths of the associated systems are determined in Ch. 13. We now have all the ingredients for comparison with the original systems T_i ($i = 0, 1, 2$) of Explicit Mathematics.

The subject matter of **Part F, Constructive, Predicative and Descriptive Mathematics in Systems of Explicit Mathematics** is obvious from its title, and Chs. 14, 15 and 16 take these developments up successively. The volume concludes with **Part G, The Operational Penumbra**, which shows how the basic operational structure may be flexibly adapted to a variety of different contexts. It is used in Ch. 17 to describe feasible operational theories, then in Ch. 18 for the so-called unfolding program for open schematic systems. Ch. 19 surveys the work on theories of universes in explicit mathematics, and the book concludes in Ch. 20 with a presentation of operational set theory. All of the chapters 17-20 are more informal, being devoted to explanations of basic notions and statements of results, detailed proofs of which can be found in the literature.

Finally, we provide an annotated comprehensive **Bibliography** of Explicit Mathematics as of the time of writing that includes all secondary references cited in the book. However, an online version of that searchable by author, date, title, and other data, will be available at http://www.iam.unibe.ch/~til/em_bibliography/ that will be maintained to update for future additions.

The book is structured so that it may be read in several different ways. For example, those with the necessary mathematical background can skip Part A while those with the needed background in the proof theory of subsystems of analysis can skip Ch. 5 of Part C. Those interested in the mathematical developments in Part F but not in the proof theory of systems of explicit mathematics can skip Part D and Ch. 13 of Part E, while those interested in the latter but not the former can simply skip Part F. In all these casa certain amount of skimming will suffice in order to go on to the main detailed developments of interest to the reader. Finally, Part G may appeal to a variety of tastes by a reading of some but not all of its chapters. Historical notes are provided throughout for those who want to learn more about the individual sources in the background literature.

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