THREE PROBLEMS FOR MATHEMATICS

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Bernays Lectures

ETH

11-12 Sept. 2012

Three Problems for Mathematics: Consistency, the Continuum, and Categories

Lecture 1: Bernays, Gödel and Hilbert's consistency program. (Tues., Sept. 11, 17:00 h)

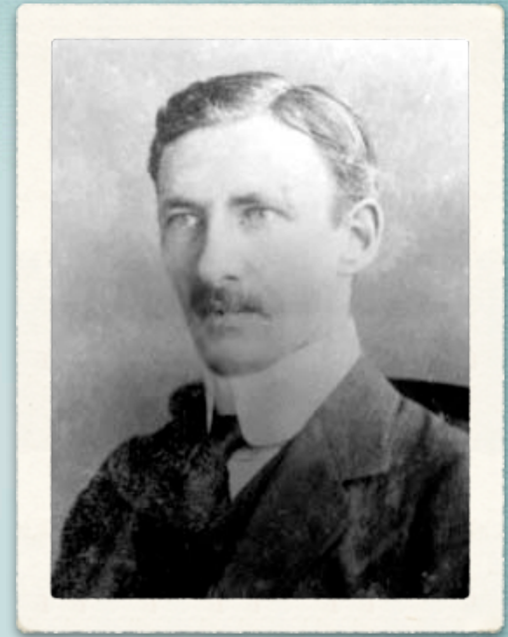
Lecture 2: Is the Continuum Hypothesis a definite mathematical problem? (Wed., Sept. 12, 14:15 h)

Lecture 3: Foundations of unlimited category theory. (Wed., Sept. 12, 16:30)

All in Auditorium C14.

David Hilbert, Paul Bernays, and Kurt Gödel

The cast of characters



Why are these Problems <u>for</u> Mathematics?

- What are the grounds for what it is legitimate to say and do in mathematics?
- They border on philosophical problems.
- They may be treated mathematically to some extent, but are not problems <u>of</u> mathematics.

Some Philosophies of Mathematics

- Platonism
- Logicism
- Constructivism
- Finitism
- Formalism

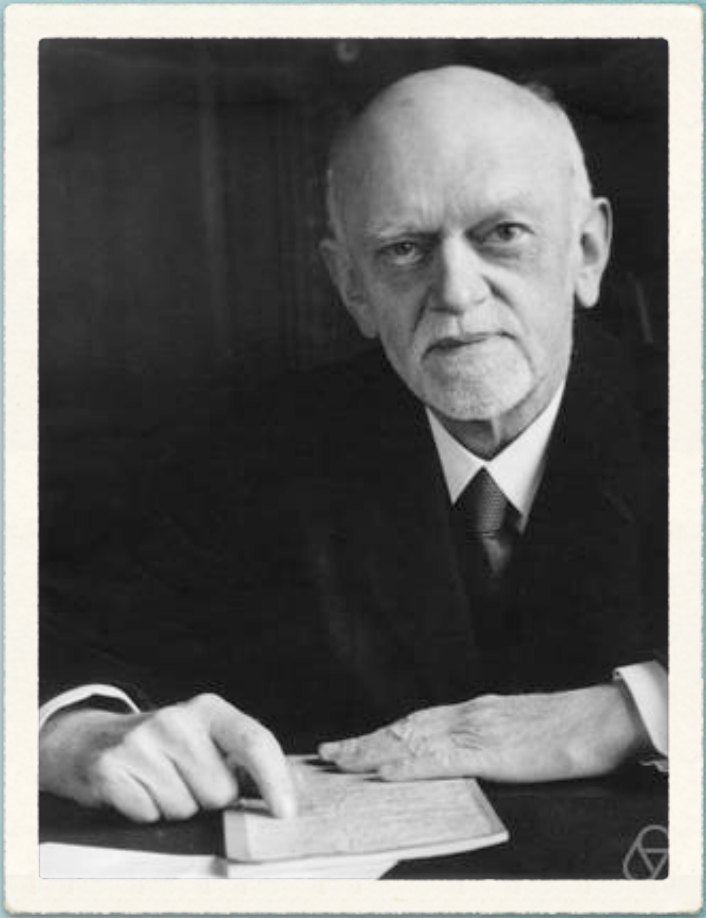
David Hilbert,
The Foremost Mathematician
of his Time

- 1862-1943
- Königsberg, Göttingen
- After Henri Poincaré, the foremost mathematician of the late19th and first third of the 20th century.
- Made fundamental contributions to all the major fields of pure and applied mathematics.

David Hilbert

1862-1943

Photo: 1932



Hilbert's Philosophy of Mathematics

- Axiomatic Foundations:
  All mathematical concepts are to be grounded axiomatically.
- A mathematical concept exists if its axioms are complete and consistent.
- This is <u>not</u> Formalism.

The Leading Foundational Axiom Systems

- Logic (Frege, Russell, Hilbert)
- Arithmetic (Peano)
- Analysis (Hilbert and Bernays)
- Set Theory (Zermelo, Fraenkel, Bernays)
- All have an "intended interpretation".

Hilbert's Finitist Consistency Program (1922-1934)

- Foundational axiom systems involving the completed infinite are problematic and must be proved consistent (analysis, set theory)
- Even arithmetic with Law of Excluded Middle (LEM) involves it:
- (for all n) P(n) or (there exists n) not-P(n).

Hilbert’s Finitist Consistency Program (continued)

- Every statement and proof in a formal axiomatic system consists of a finite sequence of symbols.
- Finitism: only reasoning about finite sequences of symbols admitted.
- Existential statements are to be witnessed.
- There is to be no use of LEM applied to universal statements.

Hilbert's Finitist Consistency Program (concluded)

- All consistency proofs are to be finitistic.
- Beweistheorie (Proof Theory, Metamathematics): the theoretical development of finitistic consistency proofs.
- Hilbert and Bernays, *Grundlagen der Mathematik*, Vol. I (1934), Vol. II (1939).

Hilbert's Shadow over Gödel

- Hilbert raised four major problems for logic and set theory (HP 1-4).
- Gödel solved all of them in full or in significant part.
- But Hilbert never acknowledged their solution or congratulated Gödel!

The Completeness of Logic Problem (HP-1)

- First-order logic (FOL): The logic of statements built from basic predicates and relations using "not", "and", "or", "implies", "for all x", "there exists x"(where x ranges over an arbitrary universe of discourse).
- Hilbert's axiomatization (Lectures1917-1918, with Bernays' assistance).

Kurt Gödel

1906-1978

Kurt Gödel

- b. Brünn, Moravia (now Brno), 1906
- Studied at the University of Vienna
- Attended meetings of the Vienna Circle
- Studied logic with Rudolf Carnap
- PhD 1929 with Hans Hahn

The Completeness of Logic Problem (HP-1, continued)

- Hilbert and Ackermann logic text, 1928: Is Hilbert's axiomatization of FOL complete?
- i.e., does a sentence A follow from Hilbert's axioms just in case it is valid in every domain of discourse?

The Completeness of Logic Problem (HP-1, continued)

- Theorem (Gödel PhD dissertation 1929). Hilbert's axiom system for FOL is complete.
- Bernays congratulated Gödel on this result in his first letter to him (1930). The beginning of an extensive correspondence.
- But Hilbert never acknowledged it. Why?

The Completeness of Arithmetic Problem (HP-2)

- Peano Axioms (PA) for the “higher arithmetic”
- Formulated in FOL. Intended universe of discourse: N = {0, 1, 2, ..., n, ...}
- Basic axioms for 0, successor, + and ×
- The Induction Axiom

The Completeness of Arithmetic Problem (HP-2, cont'd)

- Hilbert, Bologna (1928): Prove that PA is formally complete,
- i.e., show that each sentence A or its negation not-A is provable from PA.
- Theorem (Gödel1930). The extension of PA by the theory of types (PM) is not complete if it is (omega-)consistent.

Gödel's First Incompleteness Theorem (HP-2, cont'd)

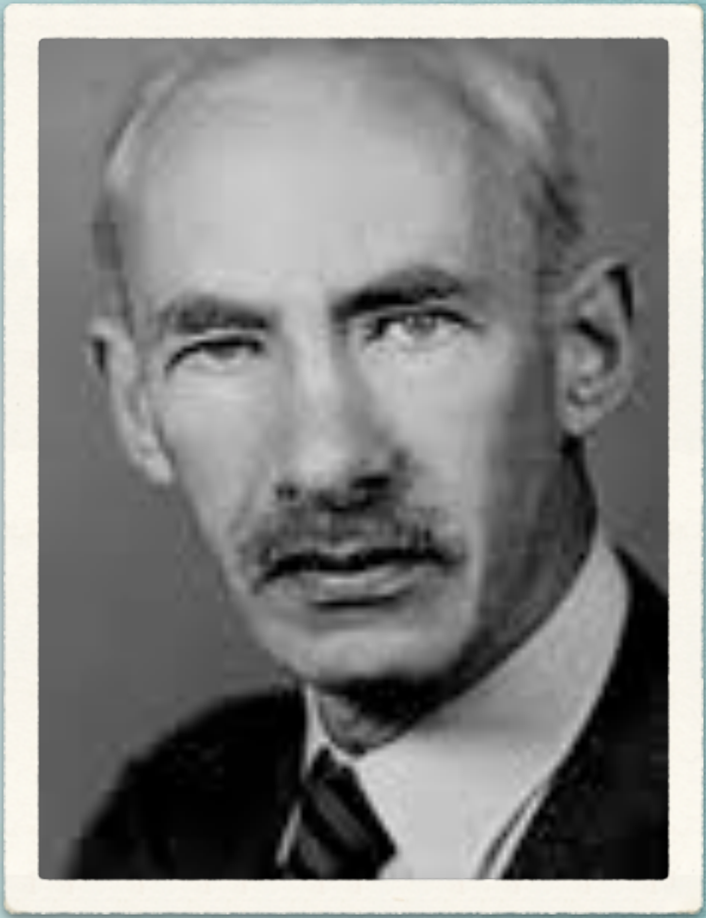
- Theorem (Gödel 1930-1931) If T is any formal axiomatic system extending PA and T is (omega-)consistent then T is incomplete.
- In fact, there are sentences A of arithmetic such that neither A nor not-A is provable in T.
- Bernays corresponded with Gödel about this but Hilbert never said a word. Why?

Paul Bernays
Between Hilbert and Gödel

- Worked with Hilbert on his logic lectures, and the formulation and exposition of his consistency program
- Fully responsible for the preparation of *Grundlagen der Mathematik*, I and II.
- Wrote Gödel in 1931 to understand his incompleteness theorems.

Paul Bernays

1888-1977



Paul Bernays

- b. London (1888), moved to Paris, Berlin
- PhD, Göttingen, 1912, under Landau in number theory
- Habilitationschrift in Zurich next year
- Hilbert's assistant in Göttingen 1917-1934

Hilbert's Program for Arithmetic (HP-3)

- (HP-3) Give a finitistic proof of the consistency of PA.
- Claimed to have been done by Ackermann, but then proof found to be faulty.

Gödel's 2nd Incompleteness Theorem and HP-3

- Theorem (Gödel 1930-1931). If T is a consistent extension of PA then the consistency of T cannot be proved in T.
- (Proved independently by von Neumann.)
- Hence, if all finitistic methods can be formalized in T then Hilbert's finitist consistency program can't be carried out for T.

Gödel's Theorem and (HP-3)

- Can all finitistic proofs be formalized in PA?
- Von Neumann--YES; Gödel, at first cautious, but within two years--YES.
- If so, (HP-3) is answered in the negative.

Hilbert's Reaction to Gödel's Second Incompleteness Theorem

- "Angry" (Bernays report)
- Incomprehension (?)
- Investment in his program
- Embarrassment

Hilbert's Reaction (cont'd)

- ...the end goal [is] to establish as consistent all our usual methods of mathematics. With respect to this goal, I would like to emphasize the following: the view, which temporarily arose and which maintained that certain recent results of Gödel show that my proof theory can't be carried out, has been shown to be erroneous.
- In fact that result shows only that one must utilize the finitary standpoint in a sharper way for the farther reaching consistency proofs… (Hilbert, *Einführung* to [Hilbert and Bernays 1934])

What are the Limits of Finitism?

- Hilbert and Bernays vague on finitism
- Gödel lectures and seminar reports, 1933, 1937: bounded by Primitive Recursive Arithmetic (PRA)--much weaker than PA.
- Hilbert: goes beyond PRA, but how far?
- The current consensus: Finitism is certainly contained in PA.

What are the Limits of Finitism? (continued)

- Gödel ,“Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes” *Dialectica* (1958)
- For Paul Bernays on his 70th birthday.
- A quantifier-free system of constructive functionals of strength PA.
- Worked on sharpenings until 1972.

Gödel unspoken Battle with Hilbert

- Gödel was a whole-hearted platonist and had no doubts about the consistency of set theory, let alone arithmetic or analysis.
- But Gödel took Hilbert's consistency program and its relativized forms seriously. Why?
- Hilbert's life-long shadow over Gödel. Gaining justice through the battle over finitism.

What's left?

- Hilbert, Gödel, and
- the Continuum Problem (HP-4)
- Tomorrow!

Reference

“Lieber Herr Bernays! Lieber Herr Gödel! Gödel on finitism, constructivity and Hilbert’s program”, in *Kurt Gödel and the Foundations of Mathematics* (M. Baaz, et al., eds.) Cambridge Univ. Press, 2011.

Previously in *Dialectica* 62 (2008), 179-203.

The End