APPROXIMATION OF VERSAL DEFORMATIONS

BRIAN CONRAD AND A.J. DE JONG

1. Introduction

In Artin's work on algebraic spaces and algebraic stacks [A2], [A3], a crucial ingredient is the use of his approximation theorem to prove the algebraizability of formal deformations under quite general conditions. The algebraizability result is given in [A2, Thm 1.6], and we recall the statement now (using standard terminology to be recalled later).

Theorem 1.1. (Artin) *Let S be a scheme locally of finite type over a field or excellent Dedekind domain, and F a contravariant functor, locally of finite presentation, from the category of S-schemes to the category of sets. Let κ be an $\mathscr{O}_S$-field of finite type, and $\xi_0 \in F(\kappa)$ an element. Assume that there exists a complete local noetherian $\mathscr{O}_S$-algebra $(\overline{A}, \mathfrak{m})$ with residue field κ and an element $\overline{\xi} \in F(\overline{A})$ lifting $\xi_0 \in F(\kappa)$ such that $\overline{\xi}$ is an effective versal deformation of ξ_0.*

Then there exists a finite type S-scheme X, a closed point $x \in X$ with residue field κ, an element $\xi \in F(X)$ lifting $\xi_0 \in F(\kappa) = F(k(x))$, and an $\mathscr{O}_S$-isomorphism $\sigma : \widehat{\mathscr{O}}_{X,x} \simeq \overline{A}$ such that $F(\sigma)(\xi)$ and $\overline{\xi}$ coincide in $F(\overline{A}/\mathfrak{m}^{n+1})$ for all $n \geq 0$. The isomorphism σ is unique if $\overline{\xi}$ is an effective universal deformation of ξ_0.

Remark 1.2. For a scheme S, an *$\mathscr{O}_S$-field of finite type* is a field κ equipped with a finite type map $\mathrm{Spec}(\kappa) \to S$. When S is locally noetherian, this is equivalent to saying that κ is a finite extension of the residue field $k(s)$ at a locally closed point $s \in S$ (see Lemma 2.1).

Whereas the techniques in [A3] are extremely conceptual and easy to digest, these methods ultimately depend upon Theorem 1.1, whose proof in [A2] (together with its clarification in [A3, Appendix]) is quite intricate and hard to "grasp". Moreover, there are two ways in which the proof of this result uses the hypothesis (harmless in practice) that S is locally of finite type over a field or excellent Dedekind domain. First, this condition is needed in Artin's original form of his approximation theorem [A1]. Second, and perhaps more seriously (in view of Popescu's subsequent proof of the Artin approximation theorem for arbitrary excellent rings), the detailed analysis in the proof of Theorem 1.1 uses very special properties of fields and Dedekind domains (such as the structure theorem for modules over a discrete valuation ring).

The restriction to base schemes locally of finite type over a field or excellent Dedekind domain in Artin's form of his approximation theorem is also the source of the restriction to finite extensions κ of residue fields at *locally closed* points of S (see Remark 1.2), rather than at general points of S, when algebraizing formal deformations as in Theorem 1.1. Since arbitrary localization preserves the property of excellence but tends to destroy the property of a map being (locally) of finite type, if one can work in the more general context of excellent base schemes then one can also hope to get algebraization results over arbitrary points of S.

In this note, we present a proof of Theorem 1.1 with S permitted to be an arbitrary excellent scheme and κ permitted to be a finite extension of $k(s)$ for an arbitrary point $s \in S$ (but of course the point $x \in X$ as in Theorem 1.1 can only be taken to be closed if and only if κ is of finite type over $\mathscr{O}_S$). In fact, we shall prove a "groupoid" generalization analogous to [A3, Cor 3.2]; see Theorem 1.5. As a consequence, the entirety of [A3] is valid as written for an arbitrary excellent base scheme S. The central technical ingredient we need is the following remarkable result of Popescu:

Date: March 13, 2002.

1991 *Mathematics Subject Classification.* Primary 14B12; Secondary 14A20.

Key words and phrases. Artin approximation, versal deformations, excellence.

This work was partially supported by the NSF and was partially conducted by the authors for the Clay Mathematics Institute. The authors would like to thank M. Artin for helpful conversations.

Theorem 1.3. (Popescu) *Let A be a noetherian ring and B a noetherian A-algebra. Then the map $A \to B$ is a regular morphism if and only if B is a direct limit of smooth A-algebras.*

Remark 1.4. Recall that a morphism $f : X \to Y$ of locally noetherian schemes is said to be *regular* if it is flat, all (locally noetherian!) fiber schemes X_y are regular, and such regularity is preserved under finite extension of the residue field (i.e., $X_y \times_{\kappa(y)} k$ is regular for any finite extension field $k/\kappa(y)$ for any $y \in Y$).

For commutative rings one makes the same definition using the corresponding affine schemes.

The most important example of Theorem 1.3 for our purposes is the case where $A \to B$ is the natural map from an excellent local ring to its completion (in which case regularity of the map is part of the definition of excellence). Popescu's proof is presented in [P1], [P2], [P3], and we refer the reader to [Sw] for a self-contained and clearly written technical exposition of the proof of Theorem 1.3 which takes into account various subsequent simplifications, and to [Sp] for a proof of a slightly stronger result than Popescu's (but proceeding along related lines). Note that although [Sw] only claims to get a filtered colimit rather than an ordinary direct limit, one can get ordinary direct limits in the main conclusion by using an easy modification of the *proof* of "$(ii) \Rightarrow (iii)$" in [L, Thm 1.2] (replacing the use of finite free modules with the use of smooth algebras); we leave the details as an exercise for the reader and we refer to [Bou, Ch X, §1.6] for the general context for such arguments. We prefer to use this stronger form of Popescu's theorem since it is psychologically simpler to understand and the condition of a functor being locally of finite presentation is typically expressed in terms of behavior with respect to direct limits (rather than more general limits); cf. [EGA, IV_3, 8.14.2]

Since the henselization of an excellent local ring A is excellent [EGA, IV_4, 18.7.3], it follows immediately from Popescu's theorem applied to $A \to \widehat{A}$ that the Artin approximation theorem is valid for any excellent local ring A, not just those essentially of finite type over a field or excellent Dedekind domain. Quite amusingly, the use of Popescu's theorem in the proof of Theorem 1.1 over an excellent base has the effect of removing the Artin approximation theorem from the proof! However the approximation theorem is necessary when considering properties of algebraizations such as "uniqueness" (as we shall see in the proof of Theorem 5.3).

The *proof* of Theorem 1.3 is intricate and technical, so one could perhaps rightly say that the input we require is more complicated than Artin's proof of Theorem 1.1. However, the *statement* of Theorem 1.3 is something which is very easy to internalize (e.g., the proof of the "if" direction is elementary), so we think it is of some interest that one can use this result to give a methodologically simpler proof of Theorem 1.1 which moreover is valid over any excellent base.

In order to give the statement of our main result, we introduce some notation. Let S be a scheme and F be a category cofibered in groupoids over the category of S-schemes. In practice, this roughly means that $F(T)$ (for a variable S-scheme T) is a category of geometric structures over T which behave well with respect to base change. For example, F could be a contravariant set-valued functor (viewed as a category with only identity map morphisms), or $F(T)$ could be the category of stable T-curves of a fixed genus or abelian T-schemes of a fixed relative dimension endowed with a polarization of a fixed degree.

For each S-scheme T we assume that the category $F(T)$ has a *set* of isomorphism class representatives, so it makes sense to define the contravariant set-valued functor $\overline{F}$ with $\overline{F}(T)$ denoting the set of isomorphism classes of objects in $F(T)$. The reader who does not like to work with categories (co)fibered in groupoids can think about the special case in which F is just a contravariant set-valued functor (so $F(T) = \overline{F}(T)$ is a small category in which the only morphisms are identity morphisms).

If κ is an $\mathscr{O}_S$-field (i.e., a field equipped with a morphism $\mathrm{Spec}(\kappa) \to S$), we say that κ is *residually finite* (over S) if $[\kappa : k(s)] < \infty$, where s is the image of $\mathrm{Spec}(\kappa) \to S$. Note that $s \in S$ can be arbitrary. Here is the main result of this paper:

Theorem 1.5. *With the notation introduced above, assume that S is excellent. Also assume that F is locally of finite presentation* (*see* (2.1)), *satisfies the Schlessinger-Rim criteria* (*see Definition* 2.5), *and the natural map of sets*

$$\overline{F}(B) \to \varprojlim \overline{F}(B/\mathfrak{m}_B^{n+1}) \tag{1.1}$$

has dense image for all complete local noetherian $\mathscr{O}_S$-algebras $(B, \mathfrak{m}_B)$ with $B/\mathfrak{m}_B$ residually finite over S.

For any residually finite $\mathscr{O}_S$-field κ and any object ξ_0 in $F(\kappa)$, any formal versal deformation of ξ_0 is algebraizable (*i.e., there exists an* (X, x) *as in Theorem* 1.1 *for our* F*, though with* x *a closed point if and only if* $s \in S$ *is a locally closed point*).

Remark 1.6. If F is instead only cofibered in groupoids over the full subcategory of locally noetherian S-schemes, the proof of Theorem 1.5 goes through without change. If we only assume the denseness of the image of (1.1) when $B/\mathfrak{m}_B$ is a finite type $\mathscr{O}_S$-field, then the conclusion of the theorem requires the extra condition that κ be a finite type $\mathscr{O}_S$-field. For most F which arise in interesting moduli problems, the map (1.1) is even bijective.

Remark 1.7. The referee called our attention to the existence of work [PR] by Popescu and Roczen from 1988 in which they assert a result which is essentially Theorem 1.5 for set-valued F (though the restriction to such F is not essential for their method). Their strategy is strikingly similar to ours, but their proof appears to be incomplete, roughly corresponding to the fact that they have no result analogous to our Theorem 3.2. More specifically, whereas we need to make approximations in a smooth algebra occuring in a direct limit process, the proof of [PR, Thm 1.2] only makes an approximation in the direct limit object itself and then argues that a certain map $g : \overline{A} \to D^\wedge$ of complete local noetherian rings is an isomorphism because it is so modulo squares of the maximal ideal. This only works if one knows the source and target rings for g have the same Hilbert series (i.e., nth order artinian quotients having the same length for all n). But there seems to be no reason to expect such an equality of Hilbert series to automatically hold in the generality suggested by the proof of [PR, Thm 1.2], and we have to argue with a refinement of the Artin-Rees lemma in order to ensure such equality.

We illustrate the gap in the proof of [PR, Thm 1.2] with a simple example (using the notation from that proof). Let $B = k$ be a field, $A = k[x]$, and $\overline{A} = k[\![x]\!]$ its x-adic completion. Let F be the functor represented by A on the category of k-algebras. We let $\xi_0 \in F(k)$ correspond to $x \mapsto 0$ and let $\xi \in F(k[\![x]\!])$ correspond to the natural map $A \to \overline{A}$ which is an algebraization of the universal deformation of ξ_0. We will now see that attempting to algebraize $(\overline{A}, \xi_0)$ following the method of proof of [PR, Thm 1.2] need not work. Since $\overline{A}$ is already a formal power series ring over k, the first step is to apply Theorem 1.3 to express $\overline{A}$ as a direct limit of smooth k-algebras and to bring down ξ through some stage of the direct limit. In general one cannot expect the smooth algebras in Theorem 1.3 to be subalgebras of the direct limit, so in our example we consider the smooth k-algebra $C = k[x, t]$ equipped with the k-map $f : C \to \overline{A}$ defined by $x \mapsto x$ and $t \mapsto 0$. Note that $t \in C$ maps to a generator of the kernel (0) of the (trivial) presentation of $\overline{A}$ as a quotient of $k[\![x]\!]$.

The obvious map $A \to C$ (defined by $x \mapsto x$) is an element $\eta \in F(C)$ for which $\xi = F(f)(\eta)$. We approximate the map f by the map $f_n : C \to k[x]$ defined by $x \mapsto x$ and $t \mapsto x^n$, and consider the artin local ring $D_n = k[x]/f_n(t) = k[x]/x^n$. The point $\eta_n \in F(D_n)$ induced by $F(f_n)(\eta)$ is just the natural quotient map $k[x] \to k[x]/x^n$. The map $\overline{A} \to \widehat{D} = D$ induced by specializing the algebraized universal deformation $\xi \in F(\overline{A})$ to the deformation $\eta_n \in F(D)$ is the natural surjection, and this is not an isomorphism no matter how large we make n (to make f_n closely approximate f). This shows that the argument in [PR] (which would use the above procedure, even with $n = 2$) is incomplete, regardless of how accurately one tries approximate f by an "algebraic" map. From the point of view of our proof of Theorem 1.5, the problem in this example is that although the base change of the C-linear presentation diagram

$$C \xrightarrow{t-x^n} C \longrightarrow C/(t-x^n) \longrightarrow 0$$

by $C \to \overline{A}$ does recover a presentation

$$\overline{A} \xrightarrow{0} \overline{A} \xrightarrow{\mathrm{id}} \overline{A} \longrightarrow 0$$

of $\overline{A} = k[\![x]\!]$, the "matrix entry" $t - x^n$ in the presentation is not at all small relative to the (x, t)-adic topology on C.

Since Theorem 1.5 renders the entirety of [A3] valid over any excellent base scheme S at all, as an immediate consequence it follows that Artin's proof of the necessary and sufficient criterion for an S-stack to be a locally finite type algebraic S-stack, given for S locally of finite type over a field or excellent Dedekind domain in [LM, Cor 10.11], is valid for any excellent base scheme S at all.

In another direction, one can ask about étale-local uniqueness of algebraizations of a given formal versal deformation. This has an affirmative answer under a mild "full faithfulness" hypothesis on F (exactly analogous to [A2, 1.7]), so one frequently has a well-defined notion of "henselized algebraization". If one assumes in addition that F is formally Deligne-Mumford (see Definition 5.5), it then makes sense to ask whether $\mathrm{Aut}_{F(\kappa)}(\xi_0)$ acts naturally on the henselized algebraization of a minimal formal versal deformation. See Theorem 5.3 and Theorem 5.7 for precise affirmative statements along these lines. Assuming in addition that $\mathrm{Aut}_{F(\kappa)}(\xi_0)$ is finite, one can even construct algebraizations to which the action of this group descends (without needing to henselize); this is formulated more precisely in Theorem 5.8.

Notation

If B is a ring and r is a non-negative integer, we denote by $B^{\oplus r}$ a finite free B-module of rank r (with specified basis). If $\varphi : M \to N$ is a B-linear map of B-modules, we denote by $\mathrm{im}(\varphi)$ the image module of φ.

If $\mathscr{C}$ is a category and X is an object in $\mathscr{C}$, we will sometimes denote this fact by writing $X \in \mathscr{C}$. This should not cause any confusion.

If S is a scheme and F is a category cofibered in groupoids over the category of S-schemes, for an affine scheme U over S we may sometimes write $F(A)$ rather than $F(U)$, where $A = \mathscr{O}_U(U)$. Also, we will write $\overline{F}$ to denote the contravariant set-valued functor defined by letting $\overline{F}(T)$ denote the set of isomorphism classes of objects in the groupoid $F(T)$.

2. General nonsense on deformations

We are concerned with the problem of approximating formal deformations by structures defined over "algebraic" rings (relative to a base). Such problems are of local nature (over the base) and hence only involve structures defined over affine base schemes, so there is no serious loss of generality in immediately focusing on functors of rings rather than functors of schemes. However, it is conceptually a bit clearer (and more geometric in spirit) to work "globally" at the start and briefly postpone the passage to the case of an affine base. In this section, we collect basic generalities along these lines for ease of reference later. The expert reader can skip ahead to §3.

Let S be a locally noetherian scheme. Following Artin, we define the category of *$\mathscr{O}_S$-algebras* to be the category of rings A equipped with a morphism $\mathrm{Spec}(A) \to S$, and we say that an $\mathscr{O}_S$-algebra A is *of finite type* if the morphism $\mathrm{Spec}(A) \to S$ is of finite type. For example, we have the basic and well-known:

Lemma 2.1. *Let κ be an $\mathscr{O}_S$-algebra which is a field, and let $s \in S$ be the image of* $\mathrm{Spec}(\kappa) \to S$. *Then κ is of finite type as an $\mathscr{O}_S$-algebra if and only if $[\kappa : k(s)]$ is finite and s is a locally closed point in S (i.e., for sufficiently small open U in S around s, the point s is closed in U).*

The example of the generic point of the spectrum of a discrete valuation ring shows that we cannot replace "locally closed" by "closed" in this lemma.

Since the property of being excellent (unlike the property of being of finite type) behaves well with respect to arbitrary localization, for our purposes it is convenient to work with a more general notion than finite type $\mathscr{O}_S$-field (though such extra generality will be harmless, since all arguments we give ultimately work with a single κ that is fixed throughout the discussion).

Definition 2.2. We shall say that an $\mathscr{O}_S$-field κ is *residually finite* (over S) if the image point s of $\mathrm{Spec}(\kappa) \to S$ is such that the degree $[\kappa : k(s)]$ is finite.

This notion certainly includes all $\mathscr{O}_S$-fields of finite type, but it allows $s \in S$ to be arbitrary.

Definition 2.3. For an $\mathscr{O}_S$-field κ lying over $s \in S$, we define $\widehat{\mathscr{C}}_S(\kappa)$ to be the category of complete local noetherian $\mathscr{O}_S$-algebras A equipped with an isomorphism $A/\mathfrak{m}_A \simeq \kappa$ over $\mathscr{O}_S$. We define $\mathscr{C}_S(\kappa)$ to be the full subcategory of artinian objects.

For the convenience of the reader, we now review some deformation-theoretic terminology. Let F be a category cofibered in groupoids over the category of S-schemes. If $\pi : B' \twoheadrightarrow B$ is a surjection of $\mathscr{O}_S$-algebras and ξ is an object in the groupoid $F(B)$, then we define a *deformation* of ξ to B' to be a pair (ξ', ι) where ξ' is an object in $F(B')$ and ι is an isomorphism from $F(\pi)(\xi')$ to ξ in $F(B)$. The deformation groupoid $F_\xi(B')$ is a subcategory of $F(B')$ defined in an evident manner (i.e., one keeps track of and demands compatibility with the ι's too). We will often suppress the mention of ι in the notation if no confusion is likely, but it is sometimes important to explicitly keep track of such extra data. The corresponding set-valued functor of isomorphism classes will be denoted $\overline{F}_{\xi_0}$ rather than the more accurate $\overline{F_{\xi_0}}$ (we will have no need to consider the deformation theory of $\overline{F}$, except of course when F is set-valued, so there is no risk of confusion with the notation $\overline{F}_{\xi_0}$).

If κ is an $\mathscr{O}_S$-field and ξ_0 is an object in $F(\kappa)$, we define the formal deformation groupoid $\widehat{F}_{\xi_0}$ on $\widehat{\mathscr{C}}_S(\kappa)$ by declaring $\widehat{F}_{\xi_0}(C)$ to be the category of projective systems (ξ_n, ι_n) with each $(\xi_n, \iota_n) \in F_{\xi_0}(C/\mathfrak{m}_C^{n+1})$.

For a pair (A, a) with $A \in \mathscr{C}_S(\kappa)$ and $a \in F_{\xi_0}(A)$, as well as a pair $(C, (\xi_n))$ with C in $\widehat{\mathscr{C}}_S(\kappa)$ and (ξ_n) in $\widehat{F}_{\xi_0}(C)$, we define

$$\mathrm{Hom}_{\widehat{F}_{\xi_0}}(\xi, a) \stackrel{\mathrm{def}}{=} \varinjlim \mathrm{Hom}_{F_{\xi_0}}(\xi_n, a),$$

where $\mathrm{Hom}_{F_{\xi_0}}(\xi_n, a)$ denotes the set of pairs (h, ψ) where

$$h : C/\mathfrak{m}_C^{n+1} \to A$$

is a map in $\mathscr{C}_S(\kappa)$ and $\psi : F(h)(\xi_n) \to a$ is an isomorphism in $F_{\xi_0}(A)$.

Definition 2.4. We say that $\xi = (\xi_n) \in \widehat{F}_{\xi_0}(C)$ is a *formal versal deformation* of ξ_0 if, for any surjection $\pi : A' \twoheadrightarrow A$ in $\mathscr{C}_S(\kappa)$ and any morphism

$$h : a' \to a$$

from $a' \in F_{\xi_0}(A')$ to $a \in F_{\xi_0}(A)$ over π, the natural map of sets

$$h \circ (\cdot) : \mathrm{Hom}_{\widehat{F}_{\xi_0}}(\xi, a') \to \mathrm{Hom}_{\widehat{F}_{\xi_0}}(\xi, a)$$

is surjective.

Our aim is to study the deformation theory of an object $\xi_0 \in F(\kappa)$, where κ is a residually finite $\mathscr{O}_S$-field and F is a "reasonable" category cofibered in groupoids over S. We make two hypotheses on F which are nearly always satisfied in practice:

- F is locally of finite presentation over S. That is, for any directed system $\{A_i\}$ of $\mathscr{O}_S$-algebras with direct limit A, the natural transformation of categories

$$\varinjlim F(A_i) \to F(A) \tag{2.1}$$

  is fully faithful and essentially surjective. This means that every object in $F(A)$ is isomorphic to the image of an object in some $F(A_i)$, and for any two objects x_i, y_i in $F(A_i)$ with corresponding induced objects $x_{i'}, y_{i'}$ in $F(A_{i'})$ (for all $i' \geq i$) and x, y in $F(A)$, the natural map of sets

$$\varinjlim \mathrm{Hom}_{F(A_{i'})}(x_{i'}, y_{i'}) \to \mathrm{Hom}_{F(A)}(x, y)$$

  is a bijection.
- For any complete local noetherian $\mathscr{O}_S$-algebra $(B, \mathfrak{m}_B)$ with $B/\mathfrak{m}_B$ residually finite over S, and any $\xi_0 \in F(B/\mathfrak{m}_B)$, the natural map of sets

$$\overline{F}_{\xi_0}(B) \to \varprojlim \overline{F}_{\xi_0}(B/\mathfrak{m}_B^{n+1}) \tag{2.2}$$

  has dense image.

The first of these two conditions is satisfied by nearly all F's which arise "in nature" [EGA, IV_3, §8ff]. The second condition is a very weak approximation hypothesis which, in practice, is nearly always an immediate consequence of Grothendieck's Existence Theorem for formal schemes [EGA, III_1, §5] and typically holds without restriction on the residue field of B. Often (2.2) is even bijective. Note that these two hypotheses on F are preserved if we restrict F to the category of S'-schemes for an S-scheme $S' \to S$ which induces finite residue field extensions at all points (e.g., S' is locally quasi-finite over S or, what is of more interest, $S' = \mathrm{Spec}(\mathscr{O}_{S,s})$ for some $s \in S$). The denseness condition in (2.2) is only to be used as a device to ensure the effectivity of formal versal deformations. As usual in deformation theory, for $\xi_0 \in F(\kappa)$ we will be particularly interested in the groupoids $F_{\xi_0}(A)$ for A in $\mathscr{C}_S(\kappa)$.

Definition 2.5. Let κ be a residually finite $\mathscr{O}_S$-field and ξ_0 an object in $F(\kappa)$. We say that F *satisfies the Schlessinger-Rim criteria at* ξ_0 if:

- F_{ξ_0} over $\mathscr{C}_S(\kappa)$ is semi-homogeneous in the sense of [SGA7, Exp VI, 1.16], which implies that the set $\overline{F}_{\xi_0}(\kappa[\varepsilon])$ admits a natural structure of κ-vector space;
- $\dim_\kappa \overline{F}_{\xi_0}(\kappa[\varepsilon]) < \infty$.

We say that F *satisfies the Schlessinger-Rim criteria* if these two properties hold for any residually finite $\mathscr{O}_S$-field κ and any object ξ_0 in $F(\kappa)$.

The Schlessinger-Rim criteria on F at ξ_0 are essentially just a groupoid version of the classical Schlessinger criteria, also allowing for the possibility that $[\kappa : k(s)] > 1$. In [SGA7, Exp VI, 1.11, 1.20], Rim adapts the techniques in [Sch] to prove that the Schlessinger-Rim criteria for F at ξ_0 are sufficient for the existence and (non-canonical) uniqueness of a *minimal* versal formal deformation $\xi \in \widehat{F}_{\xi_0}(C)$ of ξ_0. When $\kappa = k(s)$, or more generally κ is separable over $k(s)$, minimality of (C, ξ) is exactly the condition that the natural map of sets

$$\xi : \mathrm{Hom}_{\widehat{\mathscr{C}}_S(\kappa)}(C, \kappa[\varepsilon]) \to \overline{F}_{\xi_0}(\kappa[\varepsilon])$$

be bijective. When κ is allowed to be inseparable over $k(s)$, minimality involves a technical condition on $k(s)$-derivations of κ; we refer to [SGA7, Exp VI, 1.19(2)] for the precise definition in general (Rim's Λ and K' are our $\widehat{\mathscr{O}}_{S,s}$ and κ respectively, the derivation D in Rim's definition of minimality must be required to be Λ-linear, and the map γ in [SGA7, Exp VI, 1.18(2)] is K'-linear). For our purposes, the only role of Definition 2.5 will be to allow us to apply results of Rim from [SGA7, Exp VI], so it isn't necessary to provide the precise general definitions of semi-homogeneity or minimality here.

Remark 2.6. When κ is allowed to be inseparable over $k(s)$, the existence of the ring S with properties as asserted in the proof of [SGA7, Exp VI, 1.20] appears to require further justification than is given there. One can use a theorem of Grothendieck's on formal smoothness [EGA, 0_{IV}, 19.7.2] to reduce to the case $\Lambda = K$, and then induction on the inseparable degree via a slightly involved argument with exact sequences of modules of differentials takes care of the rest.

Definition 2.7. For C in $\widehat{\mathscr{C}}_S(\kappa)$ we say that an object ξ in $F_{\xi_0}(C)$ is an *effective versal deformation* of ξ_0 if the object $\widehat{\xi}$ induced by ξ in $\widehat{F}_{\xi_0}(C)$ is a formal versal deformation. If moreover $\widehat{\xi}$ is minimal, we say that ξ is a *minimal effective versal deformation*.

It is unreasonable to expect the existence of effective versal deformations unless one assumes that the maps (2.2) have dense image. In order to relate effective and formal versal deformations, we recall a standard lemma.

Lemma 2.8. *Let κ be a residually finite $\mathscr{O}_S$-field, and ξ_0 an object in $F(\kappa)$. Assume there exists a C in $\widehat{\mathscr{C}}_S(\kappa)$ and a $(\xi_n) \in \widehat{F}_{\xi_0}(C)$ which is a formal versal deformation of ξ_0. If (2.2) has dense image, then there exists an effective versal deformation in $F_{\xi_0}(C)$ which induces (ξ_n).*

Proof. See the discussion in [A2] following [A2, (1.4)]. Briefly, one first uses the denseness of the image of (2.2) to find some ξ in $F_{\xi_0}(C)$ whose image in $F_{\xi_0}(C/\mathfrak{m}_C^2)$ is isomorphic to ξ_1. Versality then gives rise to a map $\varphi : C \to C$ which induces the identity on $C/\mathfrak{m}_C^2$ and for which $\widehat{F}_{\xi_0}(\varphi)((\xi_n))$ is isomorphic to the object $\widehat{\xi}$ in $\widehat{F}_{\xi_0}(C)$ induced by ξ. Any such map φ must be surjective, and then even an automorphism, from which the lemma follows by using the object $F_{\xi_0}(\varphi^{-1})(\xi)$ in $F_{\xi_0}(C)$. ■

By Lemma 2.8, we may begin the task of algebraizing a given formal versal deformation of $\xi_0 \in F(\kappa)$ by at least assuming it to be effective. Alternatively, as in Theorem 1.1, we can simply suppose we are magically given such an effective versal deformation and abandon the hypothesis that (2.2) have dense image.

To be precise, we fix a residually finite $\mathscr{O}_S$-field κ and an object ξ_0 in $F(\kappa)$, and we fix a pair (C, ξ_C) where $C \in \widehat{\mathscr{C}}_S(\kappa)$ and $\xi_C \in F_{\xi_0}(C)$ is an effective versal deformation of ξ_0. We want to *algebraize* (C, ξ_C), which means we seek a *finite type* $\mathscr{O}_S$-algebra B equipped with residue field κ at some point $\mathfrak{p} \in \mathrm{Spec}(B)$ and an object $\xi_B \in F(B)$ such that there is an $\mathscr{O}_S$-isomorphism $\widehat{B}_{\mathfrak{p}} \simeq C$ respecting residue field identifications with κ and carrying ξ_B to an object in $F(C)$ isomorphic to ξ_C.

Note that it suffices to work with B's which are local and essentially finite type over $\mathscr{O}_S$, since ξ_B can then always be "smeared out" over a finite type $\mathscr{O}_S$-algebra (as F is locally of finite presentation). In such a local ring situation we will speak of "local algebraizations" to avoid abuse of terminology (as algebraizations are supposed to be of finite type over S); the distinction is minor since F is locally of finite presentation. We will carry out a construction of a local algebraization (B, ξ_B) under the hypothesis that S is *excellent*.

Since these approximation problems only depend on a neighborhood of s in S, and even just depend on situations over the excellent affine scheme $\mathrm{Spec}(\mathscr{O}_{S,s})$ in which s is a closed point, there is no loss of generality in now passing to fibered categories over a category of rings and assuming that s is a *closed* point in an *affine* S (cf. Lemma 2.1). We shall adopt this point of view from now on.

3. Some algebra

We need a couple of elementary lemmas which center on the Artin-Rees lemma. The main point is to control the constant which arises in the Artin-Rees lemma when one approximates a given linear map by other linear maps.

Let A be a noetherian ring and $\mathfrak{m}$ an ideal in A. In subsequent applications A will be a complete local ring and $\mathfrak{m}$ will be its maximal ideal, but we do not need such conditions here. By the Artin-Rees lemma, if

$$X : A^{\oplus r_1} \to A^{\oplus r_2}$$

is a map of finite free A-modules, then there exists a non-negative integer c such that

$$X(A^{\oplus r_1}) \cap \mathfrak{m}^n A^{\oplus r_2} \subseteq X(\mathfrak{m}^{n-c} A^{\oplus r_1}) \tag{3.1}$$

for all $n \geq c$. We summarize this situation by saying "c works for $(X, \mathfrak{m})$ in the Artin-Rees lemma". Note that we can always increase such a c without affecting that it "works". Of course, we could avoid mentioning explicit bases of finite free modules (and could even work with finite locally free modules), but it simplifies the exposition to use matrix and vector notation and this is harmless for the subsequent applications.

Lemma 3.1. *Let A be a noetherian ring and $\mathfrak{m}$ an ideal in A. Let*

$$A^{\oplus r_0} \xrightarrow{Y} A^{\oplus r_1} \xrightarrow{X} A^{\oplus r_2} \tag{3.2}$$

be an exact complex of finite free A-modules. Let c work for both $(Y, \mathfrak{m})$ and $(X, \mathfrak{m})$ in the Artin-Rees lemma. Consider a pair of matrices Y' and X' such that

$$A^{\oplus r_0} \xrightarrow{Y'} A^{\oplus r_1} \xrightarrow{X'} A^{\oplus r_2} \tag{3.3}$$

is a complex and such that

$$Y \equiv Y' \bmod \mathfrak{m}^{c+1}, \quad X \equiv X' \bmod \mathfrak{m}^{c+1}.$$

Then we have the following conclusions:

1. *The* same *constant c works for $(X', \mathfrak{m})$ in the Artin-Rees lemma.*
2. *The complex* (3.3) *is exact; i.e.,* $\ker(X') = \mathrm{im}(Y')$.

Later on we will be interested in such diagrams with $r_2 = 1$, in which case (3.2) will arise from a presentation of a quotient algebra of A. The focus of interest will then be on showing that we can "deform" X to X' without changing the constant which works in the Artin-Rees lemma. Keeping track of an Artin-Rees constant for Y is what enables one to get such precise control for X'. The second part of the lemma will never be used in what follows.

Proof. By localizing, we may assume that A is local. The case $\mathfrak{m} = A$ is trivial, so we may assume that $\mathfrak{m}$ lies inside of the maximal ideal of A. In particular, all finite A-modules are $\mathfrak{m}$-adically separated.

We begin by proving the first part of the lemma. Fix $n \geq c$ and an integer $\ell \geq 0$. Choose a vector $\vec{a} \in \mathfrak{m}^\ell A^{\oplus r_1}$ such that

$$X'(\vec{a}) \in \mathfrak{m}^n A^{\oplus r_2}.$$

We will show by descending induction on ℓ that $X'(\vec{a}) = X'(\vec{\alpha})$ for some $\vec{\alpha} \in \mathfrak{m}^{n-c}A^{\oplus r_1}$, thereby obtaining that c works for $(X', \mathfrak{m})$ in the Artin-Rees lemma.

If $\ell \geq n - c$ there is nothing to show, so we may assume $\ell < n - c$. We will find an element $Y'(\vec{b})$ which differs from $\vec{a}$ by an element of $\mathfrak{m}^{\ell+1}A^{\oplus r_1}$. Once this is done, we can replace $\vec{a}$ with $\vec{a} - Y'(\vec{b})$ without affecting $X'(\vec{a})$ but increasing ℓ by 1 in the process, so the first part of the lemma would follow.

Note that

$$X(\vec{a}) = X'(\vec{a}) + (X - X')(\vec{a}) \in (\mathfrak{m}^n + \mathfrak{m}^{c+1+\ell})A^{\oplus r_2} = \mathfrak{m}^{c+1+\ell}A^{\oplus r_2}$$

since $\ell < n - c$. Since c works in the Artin-Rees lemma for $(X, \mathfrak{m})$, by using $n = c + 1 + \ell$ in (3.1) we get

$$\vec{a} \in \mathfrak{m}^{\ell+1}A^{\oplus r_1} + \ker(X) = \mathfrak{m}^{\ell+1}A^{\oplus r_1} + \mathrm{im}(Y)$$

since $\ker(X) = \mathrm{im}(Y)$ by the exactness of (3.2). In other words,

$$\vec{a} = Y(\vec{b}) + \vec{a}_1$$

for some $\vec{b} \in A^{\oplus r_0}$ and some $\vec{a}_1 \in \mathfrak{m}^{\ell+1}A^{\oplus r_1}$. Writing this as

$$Y(\vec{b}) = \vec{a} - \vec{a}_1 \in \mathfrak{m}^{\ell}A^{\oplus r_1}$$

and recalling that c works in the Artin-Rees lemma for $(Y, \mathfrak{m})$, we can choose $\vec{b}$ so that

$$\vec{b} \in \mathfrak{m}^{\max\{0, \ell - c\}}A^{\oplus r_0}.$$

In particular, since $Y \equiv Y' \bmod \mathfrak{m}^{c+1}$ we conclude that

$$(Y - Y')(\vec{b}) \in \mathfrak{m}^{\ell+1}A^{\oplus r_1}$$

by treating separately the cases $\ell > c$ and $\ell \leq c$.

Thus,

$$\vec{a} = Y(\vec{b}) + \vec{a}_1 = Y'(\vec{b}) + (Y - Y')(\vec{b}) + \vec{a}_1$$

with the last two addends in $\mathfrak{m}^{\ell+1}A^{\oplus r_1}$. If we replace $\vec{a}$ with $\vec{a} - Y'(\vec{b})$, then the image $X'(\vec{a})$ is unaffected but ℓ goes up by 1. This completes the proof of the first part of the lemma, but since the method of proof only involves subtracting off well-chosen elements in the image of Y' at every step, we get the slightly stronger conclusion:

$$X'^{-1}(\mathfrak{m}^n A^{\oplus r_2}) \subseteq \mathrm{im}(Y') + \mathfrak{m}^{n-c}A^{\oplus r_1} \tag{3.4}$$

This holds for all $n \geq c$.

Now we use the $\mathfrak{m}$-adic separatedness of finite A-modules. Forming intersections of (3.4) over all $n \geq c$, we get

$$\ker(X') = X'^{-1}(0) = X'^{-1}\left(\bigcap_{n \geq c} \mathfrak{m}^n A^{\oplus r_2}\right) \subseteq \bigcap_{n \geq c} (\mathrm{im}(Y') + \mathfrak{m}^{n-c}A^{\oplus r_1}) = \mathrm{im}(Y') \subseteq \ker(X'),$$

the final inclusion because (3.3) is a complex. This gives the assertion that the complex formed by Y' and X' is exact.

■

With A and $\mathfrak{m}$ as above, we define the graded noetherian ring

$$\mathrm{Gr}_{\mathfrak{m}}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$$

and for any A-module M we define the graded $\mathrm{Gr}_{\mathfrak{m}}(A)$-module

$$\mathrm{Gr}_{\mathfrak{m}}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M / \mathfrak{m}^{n+1} M.$$

The control on the constant in the Artin-Rees lemma from Lemma 3.1 is used to prove:

Theorem 3.2. *Let X, X', Y, Y' be as in Lemma 3.1. There is a unique isomorphism of graded $\mathrm{Gr}_{\mathfrak{m}}(A)$-modules*

$$\mathrm{Gr}_{\mathfrak{m}}(\mathrm{coker}(X)) \simeq \mathrm{Gr}_{\mathfrak{m}}(\mathrm{coker}(X')) \tag{3.5}$$

as quotients of $\mathrm{Gr}_{\mathfrak{m}}(A^{\oplus r_2})$. When $r_2 = 1$, so the images of X and X' are ideals in A, then the isomorphism (3.5) is one of graded $\mathrm{Gr}_{\mathfrak{m}}(A)$-algebras.

In later applications we will be in cases with $r_2 = 1$. The importance of Theorem 3.2 will then be that "slightly" modifying the relations in an A-algebra quotient of A will not affect the associated graded algebra of its maximal-adic completion as an algebra over the residue field. This uniform control at all levels of the maximal-adic filtration is the means by which we will be able to conclude that certain "algebraic" structures recover given formal structures upon passage to completions (see the use of (4.13) to prove that (4.15) is an isomorphism).

Although the main theorem in [Eis] gives a result quite similar in appearance to Theorem 3.2, the torsion condition on homology in that theorem seems to render [Eis] inapplicable in our present situation. More specifically, one might try to apply [Eis] to our situation modulo $\mathfrak{m}^t$ for $t = 1, 2, \ldots$, but then the order of approximation arising from [Eis] would a priori depend on t. It may be possible via Lemma 3.1(1) (which does not seem to follow from [Eis]) to use [Eis] to prove Theorem 3.2, but this would require unwinding the role of the Artin-Rees lemma in a lot of spectral sequences and so at best seems likely to be more technical and longer than the arguments we give (if such an alternative argument can be found).

Proof. For any $n \geq 0$ we have

$$\mathrm{Gr}_{\mathfrak{m}}(\mathrm{coker}(X)) = \bigoplus_{n \geq 0} \mathfrak{m}^n A^{\oplus r_2} / (\mathfrak{m}^{n+1} A^{\oplus r_2} + \mathfrak{m}^n A^{\oplus r_2} \cap \mathrm{im}(X)),$$

and likewise for X' replacing X. Thus, what we need to show is

$$\mathfrak{m}^{n+1} A^{\oplus r_2} + \mathfrak{m}^n A^{\oplus r_2} \cap \mathrm{im}(X) = \mathfrak{m}^{n+1} A^{\oplus r_2} + \mathfrak{m}^n A^{\oplus r_2} \cap \mathrm{im}(X') \tag{3.6}$$

for all $n \geq 0$. If $n \leq c$ (with c as in Lemma 3.1) then the hypothesis $X \equiv X' \bmod \mathfrak{m}^{c+1}$ yields (3.6). Thus, we can (and do) now focus our attention on cases with $n > c$.

Since c works in the Artin-Rees lemma for $(X, \mathfrak{m})$, we see that

$$\mathfrak{m}^n A^{\oplus r_2} \cap \mathrm{im}(X) \subseteq X(\mathfrak{m}^{n-c} A^{\oplus r_1}).$$

But c also works in the Artin-Rees lemma for $(Y, \mathfrak{m})$, and hence for $(X', \mathfrak{m})$ too (by the first part of Lemma 3.1), so

$$\mathfrak{m}^n A^{\oplus r_2} \cap \mathrm{im}(X') \subseteq X'(\mathfrak{m}^{n-c} A^{\oplus r_1}).$$

When $\vec{a} \in \mathfrak{m}^{n-c} A^{\oplus r_1}$ we have

$$(X - X')(\vec{a}) \in \mathfrak{m}^{c+1} \mathfrak{m}^{n-c} A^{\oplus r_2} = \mathfrak{m}^{n+1} A^{\oplus r_2},$$

(the addition/subtraction of which is therefore harmless for detecting membership in either side of (3.6)) and hence for such $\vec{a}$ we trivially have

$$X(\vec{a}) \in \mathfrak{m}^n A^{\oplus r_2} \Leftrightarrow X'(\vec{a}) \in \mathfrak{m}^n A^{\oplus r_2}.$$

Putting these observations together, we get (3.6) for $n > c$. ■

4. Approximation for groupoids

We are now ready to give the proof of Theorem 1.5. Before starting the proof, we should remark that the proof of [A3, Cor 3.2] asserts that the special case of Theorem 1.5 for S as in Theorem 1.1 and finite type $\mathscr{O}_S$-fields κ follows purely formally from the statement of the "set-valued" analogue Theorem 1.1 applied to the set-valued functor $\overline{F}$. This seems not quite accurate: we need to keep track of isomorphisms when algebraizing a given formal deformation, and hence passing to $\overline{F}$ in the algebraization process appears to cause too much loss of information. Partly for this reason, we need to work throughout with stacks fibered in groupoids rather than with set-valued functors in order to prove Theorem 1.5.

As we explained at the end of §2, there is no loss of generality in restricting to a category fibered in groupoids over a category of rings rather than a category cofibered in groupoids over a category of schemes. More specifically, we let R be an *excellent* ring and F a category fibered in groupoids over the the category of R-algebras. We assume that F is locally of finite presentation. We pick a complete local noetherian R-algebra C with residue field $\kappa = \kappa(C)$ of finite degree over the residue field $k = R/\mathfrak{m}$ at a *maximal* ideal $\mathfrak{m}$ of R, and we assume that we are given $\xi_C \in F(C)$ which is an effective versal deformation of

$$\xi_0 \stackrel{\text{def}}{=} \xi_C \bmod \mathfrak{m}_C \in F(\kappa).$$

Our aim is to "local-algebraize" the pair (C, ξ_C) in the sense discussed at the end of §2.

Since the residue field κ of C is finite over the residue field $k = R/\mathfrak{m}$ at a maximal ideal of R, we can find an R-algebra map

$$\varphi : R[t_1, \ldots, t_s] \to C$$

such that $\mathfrak{m}' \stackrel{\text{def}}{=} \varphi^{-1}(\mathfrak{m}_C)$ is a maximal ideal of $R[t_1, \ldots, t_s]$ and the natural local map

$$A \stackrel{\text{def}}{=} R[t_1, \ldots, t_s]^{\wedge}_{\mathfrak{m}'} \to C \tag{4.1}$$

is *surjective*. Define

$$B = R[t_1, \ldots, t_s]_{\mathfrak{m}'},$$

so A is the maximal-adic completion of B. We choose a free resolution of the A-module C

$$A^{\oplus r_0} \xrightarrow{Y} A^{\oplus r_1} \xrightarrow{X} A \longrightarrow C \tag{4.2}$$

where the last map sends $1 \mapsto 1$. We are going to use Popescu's Theorem 1.3 to approximate this "completed" situation using essentially finite type B-algebras (which are of course also essentially finite type R-algebras). The isomorphism in Theorem 3.2 will provide adequate control on maximal-adic filtrations to ensure that our essentially finite type analogue of (4.2) does in fact recover (C, ξ_C) upon completion.

Since R is excellent, so B is an excellent local ring, the map $B \to \widehat{B} = A$ is a *regular* morphism. Thus, by Theorem 1.3 we can write

$$A \simeq \varinjlim B_\lambda \tag{4.3}$$

for a directed system $\{B_\lambda\}$ of essentially *smooth* local B-algebras (i.e., each B_λ is a local ring at a point on a smooth B-scheme, with all transition maps local as well). Note that all ring homomorphisms

$$B \to B_\lambda \to B_{\lambda'} \to A \to C$$

are not only local but even induce isomorphisms on residue fields (since $B \to C$ induces an isomorphism on residue fields, as $A = \widehat{B}$ and (4.1) is surjective).

Applying standard direct limit arguments to (4.2) with the help of (4.3), for a sufficiently large λ_0 we can find matrices Y_{λ_0} and X_{λ_0} over B_{λ_0} inducing a *complex* of B_{λ_0}-linear maps

$$B_{\lambda_0}^{\oplus r_0} \xrightarrow{Y_{\lambda_0}} B_{\lambda_0}^{\oplus r_1} \xrightarrow{X_{\lambda_0}} B_{\lambda_0} \tag{4.4}$$

which recovers the first two maps in (4.2) upon applying the extension of scalars $B_{\lambda_0} \to A$. For $\lambda \geq \lambda_0$ we define

$$B_{\lambda}^{\oplus r_0} \xrightarrow{Y_{\lambda}} B_{\lambda}^{\oplus r_1} \xrightarrow{X_{\lambda}} B_{\lambda} \tag{4.5}$$

to be the extension of scalars of (4.4) by $B_{\lambda_0} \to B_\lambda$, so the first two maps in the diagram (4.2) constitute the direct limit of the diagrams (4.5) over $\lambda \geq \lambda_0$. In what follows, we implicitly suppose all subscripts λ satisfy $\lambda \geq \lambda_0$.

The cokernels

$$C_\lambda = \operatorname{coker}(X_\lambda) \tag{4.6}$$

form a directed system of B_λ-*algebras* over the directed system $\{B_\lambda\}$, compatible with base changes by $B_\lambda \to B_{\lambda'}$ for $\lambda' \geq \lambda$. Moreover, since the right-hand map in (4.2) sends $1 \mapsto 1$, we have a natural map of $\varinjlim B_\lambda = B$-*algebras*

$$\varinjlim C_\lambda \to C$$

which is visibly an isomorphism. Since F is locally of finite presentation, we can therefore find a large λ_1 and an object

$$\xi_{\lambda_1} \in F(C_{\lambda_1})$$

admitting a map $\xi_{\lambda_1} \to \xi_C$ over $C_{\lambda_1} \to C$.

Recalling that $A = \widehat{B}$, we have the following commutative diagram of local maps of local rings:

(4.7)
$$\begin{array}{ccccc} & \nearrow & A & \longrightarrow & C \\ & & \uparrow{\scriptstyle j} & & \uparrow \\ B & \longrightarrow & B_{\lambda_1} & \longrightarrow & C_{\lambda_1} \\ & \nwarrow & \uparrow & \nearrow & \\ & & R & & \end{array}$$

where the two right horizontal maps are surjections (cf. (4.1) and (4.6)) and the right-most vertical map underlies a map $\xi_{\lambda_1} \to \xi$. In particular, we have a commutative diagram of completions

(4.8)
$$\underbrace{A = \widehat{B} \longrightarrow \widehat{B}_{\lambda_1} \xrightarrow{\widehat{j}} A}_{1_A}$$

Moreover, since $A \to C$ is surjective and $\widehat{B} \to A$ is an isomorphism, it follows that the maximal ideal of B generates that of C, so the commutativity of (4.7) shows that the maximal ideal of C_{λ_1} generates that of C. We will use the notation $\mathfrak{n}$ for the "common" maximal ideal of C_{λ_1} and C.

Lemma 4.1. *There exist* $y_1, \ldots, y_r \in \widehat{B}_{\lambda_1}$ *such that there is an isomorphism*

$$\widehat{B}_{\lambda_1} \simeq A[\![y_1, \ldots, y_r]\!]$$

compatible with (4.8). *In particular, this isomorphism respects the* R*-algebra structures on both sides.*

Proof. This follows from the fact that $A = \widehat{B}$ and the local map $B \to B_{\lambda_1}$ is essentially smooth with trivial residue field extension (cf. proof of [EGA, IV_4, 17.5.3]). ∎

Now fix a large integer $N \geq 2$ (to be determined later using the Artin-Rees lemma for (4.2)) and consider $x_1, \ldots, x_r \in B_{\lambda_1}$ such that

(4.9)
$$x_i \equiv y_i \bmod \mathfrak{m}^N_{\widehat{B}_{\lambda_1}}$$

for all i. We write

$$\overline{B}_{\lambda_1} = B_{\lambda_1}/(x_1, \ldots, x_r)$$

and we define $\overline{B}_{\lambda_1}$-linear maps $\overline{X}_{\lambda_1}$, $\overline{Y}_{\lambda_1}$ accordingly by using (4.5) for $\lambda = \lambda_1$. By Lemma 4.1 and (4.9), since $N \geq 2$ we see that the natural map

(4.10)
$$\psi : A = \widehat{B} \to \overline{B}^{\wedge}_{\lambda_1} \simeq \widehat{B}_{\lambda_1}/(x_1, \ldots, x_r) \simeq A[\![y_1, \ldots, y_r]\!]/(x_1, \ldots, x_r)$$

induced by passage to the completion on $B \to \overline{B}_{\lambda_1}$ is an isomorphism.

Since ψ is an isomorphism, it follows that the map $B \to \overline{B}_{\lambda_1}$ is essentially étale [EGA, IV_4, 17.6.3]. In addition, we see with the help of Lemma 4.1 and (4.9) that ψ has the following crucial property: for all $b \in \widehat{B}_{\lambda_1} \simeq A[\![y_1, \ldots, y_r]\!]$,

$$\widehat{j}(b) - \psi^{-1}(\overline{b}) \in \mathfrak{m}^N_A,$$

where $\overline{b} = b \bmod (x_1, \dots, x_r) \in \overline{B}^{\wedge}_{\lambda_1}$.

Consequently, the matrices

$$X' \overset{\text{def}}{=} \psi^{-1}(\overline{X}^{\wedge}_{\lambda_1}), \quad Y' \overset{\text{def}}{=} \psi^{-1}(\overline{Y}^{\wedge}_{\lambda_1})$$

yield an A-linear complex

$$A^{\oplus r_0} \xrightarrow{Y'} A^{\oplus r_1} \xrightarrow{X'} A \longrightarrow C' \tag{4.11}$$

(with $1 \mapsto 1$ on the right) which is congruent to (4.2) modulo $\mathfrak{m}_A^N$.

Let us write

$$\overline{C}_{\lambda_1} \overset{\text{def}}{=} C_{\lambda_1}/(x_1, \dots, x_r) = \operatorname{coker}(\overline{X}_{\lambda_1} : \overline{B}^{\oplus r_1}_{\lambda_1} \to \overline{B}_{\lambda_1}).$$

Beware that $\overline{C}_{\lambda_1}$ is "uncompleted", so it has no A-algebra structure. As we noted before Lemma 4.1, C_{λ_1} and C have the "same" maximal ideal $\mathfrak{n}$. This notation "$\mathfrak{n}$" will also be used for the maximal ideal of the quotient $\overline{C}_{\lambda_1}$ of C_{λ_1}. Using the $\widehat{B} = A$-algebra isomorphism

$$\overline{C}^{\wedge}_{\lambda_1} \simeq C' \overset{\text{def}}{=} \operatorname{coker}(X') \tag{4.12}$$

defined via the isomorphism ψ, by Theorem 3.2 we obtain a graded $R/\mathfrak{m}$-algebra isomorphism

$$\mathrm{Gr}_{\mathfrak{n}} \overline{C}_{\lambda_1} \simeq \mathrm{Gr}_{\mathfrak{n}} C \tag{4.13}$$

if we take N in (4.9) to be large enough (depending *only* on the Artin-Rees lemma for the matrices X and Y in (4.2) to make (4.11) sufficiently congruent to (4.2)). In particular, for *every* non-negative integer i, the ith graded pieces of the graded $R/\mathfrak{m}$-algebras in (4.13) have the *same* (finite) dimension over $R/\mathfrak{m}$.

Let $\overline{\xi}_{\lambda_1} \in F(\overline{C}_{\lambda_1})$ be the object obtained from $\xi_{\lambda_1} \in F(C_{\lambda_1})$ via the quotient map $C_{\lambda_1} \to \overline{C}_{\lambda_1}$. Passing to the quotient of the isomorphism (4.10) by the Nth powers of the maximal ideals and using (4.12), we have an identification as C_{λ_1}-algebras between

$$C/\mathfrak{n}^N = \operatorname{coker}(X) \bmod \mathfrak{m}_A^N \simeq \operatorname{coker}(X') \bmod \mathfrak{m}_A^N$$

and

$$\overline{C}_{\lambda_1}/\mathfrak{n}^N = \operatorname{coker}(\overline{X}_{\lambda_1}) \bmod \mathfrak{m}^N_{\overline{B}_{\lambda_1}}.$$

That is, we get an R-algebra isomorphism $C/\mathfrak{n}^N \simeq \overline{C}_{\lambda_1}/\mathfrak{n}^N$ which makes the diagram

$$\begin{array}{ccccc} C_{\lambda_1} & \longrightarrow & C & \longrightarrow & C/\mathfrak{n}^N \\ & \searrow & & & \downarrow \simeq \\ & & \overline{C}_{\lambda_1} & \longrightarrow & \overline{C}_{\lambda_1}/\mathfrak{n}^N \end{array} \tag{4.14}$$

commute. In particular, the pushfoward $\overline{\xi}^{\wedge}_{\lambda_1} \in F(\overline{C}^{\wedge}_{\lambda_1})$ of $\overline{\xi}_{\lambda_1} \in F(\overline{C}_{\lambda_1})$ is a deformation of $\xi_0 \in F(\kappa)$.

The hypothesis that (C, ξ_C) is an effective versal deformation of ξ_0 implies that there is a local map of local R-algebras (respecting the residue field κ)

$$\sigma : C \to \overline{C}^{\wedge}_{\lambda_1} \tag{4.15}$$

which lifts the right-hand column of (4.14) and is such that $F(\sigma)(\xi_C)$ and $\overline{\xi}^{\wedge}_{\lambda_1}$ are isomorphic in $\widehat{F}_{\xi_0}(\overline{C}^{\wedge}_{\lambda_1})$. Since $N \geq 2$ and σ is an isomorphism modulo Nth powers of the maximal ideals, it follows that σ is at least *surjective*. But recall from our above analysis of (4.13) and addivity of length that the quotients $C/\mathfrak{n}^M$ and $\overline{C}_{\lambda_1}/\mathfrak{n}^M$ have the same finite R-length for each $M \geq 1$. Hence, any R-linear surjection between these must be an isomorphism. It follows that the R-algebra surjection σ must be an isomorphism. The pair $(\overline{C}_{\lambda_1}, \overline{\xi}_{\lambda_1})$ is the desired local algebraization of the initial pair (C, ξ_C). This completes the proof of Theorem 1.5 (note that one can only get $x \in X$ in Theorem 1.5 to be a closed point precisely when κ is of finite type over $\mathscr{O}_S$).

5. Some properties of algebraizations

There are two additional questions one can ask concerning Theorem 1.5:

- Is the algebraization of a given formal versal deformation étale-locally unique?
- If so, does the group $\mathrm{Aut}_{F(\kappa)}(\xi_0)$ canonically act on the "henselization" of an algebraization in the minimal versal case?

We will make these questions precise in a moment. It scarcely makes sense to think about either of these questions unless the natural transformation

$$F(B) \to \widehat{F}(\widehat{B}) \stackrel{\mathrm{def}}{=} \varprojlim F(B/\mathfrak{m}_B^{n+1}) \tag{5.1}$$

is faithful for every local noetherian $\mathscr{O}_S$-algebra $(B, \mathfrak{m}_B)$ with residually finite $B/\mathfrak{m}_B$, and is fully faithful when B is also *complete*; the right side of (5.1) denotes the evident groupoid of projective systems. In practice, the faithfulness for noetherian local B and the full faithfulness for complete noetherian local B follow from fpqc descent theory (see [EGA, IV_2, §2.5–2.6] and [BLR, §6.1]) for $\mathrm{Spec}(\widehat{B}) \to \mathrm{Spec}(B)$ and Grothendieck's formal GAGA theorems [EGA, III_1, §5] over $\widehat{B}$. Note that such full faithfulness in the complete case ensures that if (B_1, ξ_1) and (B_2, ξ_2) are effective minimal versal deformations of ξ_0, then there exists an isomorphism $B_1 \simeq B_2$ in $\widehat{\mathscr{C}}_S(\kappa)$ and a compatible isomorphism $\xi_1 \simeq \xi_2$. This is immediate from the (non-canonical) uniqueness of minimal formal versal deformations. Thus, it is meaningful to ask about "uniqueness" for algebraizations of minimal formal versal deformations, or more generally of a fixed formal versal deformation.

Definition 5.1. We say that F is *formally faithful* if (5.1) if faithful for local noetherian $\mathscr{O}_S$-algebras B with residually finite residue field $B/\mathfrak{m}_B$ and is fully faithful for such complete B.

Definition 5.2. If $(\overline{C}_1, \xi_{\overline{C}_1})$ and $(\overline{C}_2, \xi_{\overline{C}_2})$ are two local algebraizations of an effective versal deformation (C, ξ_C) of ξ_0, we say that they are *strictly étale-locally isomorphic* if they become isomorphic upon pullback to some common local-étale extension of the $\overline{C}_j$'s with trivial residue field extension on κ.

Since F is locally of finite presentation, it is easy to see that Definition 5.2 is equivalent to the analogous property for (non-local) algebraizations in terms of ordinary (residually trivial) étale neighborhoods of the base point. In more canonical terms, Definition 5.2 demands the existence of an $\mathscr{O}_S$-algebra isomorphism

$$\overline{C}_1^{\mathrm{h}} \simeq \overline{C}_2^{\mathrm{h}} \tag{5.2}$$

of henselizations which lifts the identity on the residue field κ and lies under an isomorphism between $\xi_{\overline{C}_1}$ and $\xi_{\overline{C}_2}$ as deformations of ξ_0. Here is the affirmative result concerning the existence of such an isomorphism (this is essentially just [A2, 1.7] adapted to our setting):

Theorem 5.3. *Under the hypotheses of Theorem* 1.5*, assume also that F is formally faithful. Choose a residually finite $\mathscr{O}_S$-field κ and an object ξ_0 in $F(\kappa)$. If $(\xi_{\overline{C}_1}, \overline{C}_1)$ and $(\xi_{\overline{C}_2}, \overline{C}_2)$ are two algebraizations of an effective versal deformation (C, ξ_C) of ξ_0, then an isomorphism as in* (5.2) *exists.*

Remark 5.4. This theorem permits us to speak of a "henselized algebraization" of a fixed formal versal deformation, though such data is only unique up to non-canonical isomorphism in general.

Proof. Since F is formally faithful, so (5.1) is fully faithful for complete B, the algebraization property implies that the isomorphisms $C \simeq \overline{C}_j^{\wedge}$ carry ξ_C over to an object in $F_{\xi_0}(\overline{C}_j^{\wedge})$ which is isomorphic to the pushfoward $\xi_{\overline{C}_j}^{\wedge}$ of $\xi_{\overline{C}_j}$ in $F_{\xi_0}(\overline{C}_j^{\wedge})$ (and not just its pushfoward in $\widehat{F}_{\xi_0}(\overline{C}_j^{\wedge})$).

Artin's use of "$\xi_C \mapsto \xi_{\overline{C}_j}^{\wedge}$" to construct the desired isomorphism (5.2) in the proof of [A2, 1.7] carries over essentially verbatim to the present setting of an arbitrary excellent base. Although [A2] only works with set-valued functors rather than fibered categories, the argument in [A2, pp. 32–33] adapts easily to the case of our locally finitely presented fibered category F. The only step requiring a slight modification is where Artin appeals to the Artin approximation theorem: we need to use Popescu's Theorem 1.3 to provide the required generalization of Artin approximation to our present setting of an arbitrary excellent base scheme (rather than one locally of finite type over a field or excellent Dedekind domain).

■

We now turn to the other question raised above: can we make automorphisms of ξ_0 naturally act on the "henselized algebraization" of a formal versal deformation, at least when the corresponding formal versal deformation is minimal and the deformation functor F_{ξ_0} is "set-valued" (even when F is not)?

For example, let $S = \mathrm{Spec}(\mathbf{Z}_{(2)}^{\mathrm{sh}})$ be a strict henselization of $\mathrm{Spec}(\mathbf{Z}_{(2)})$ with residue field $\kappa = \overline{\mathbf{F}}_2$ at the closed point, let E be a supersingular elliptic curve over κ, and let the elliptic curve $\mathscr{E} \to \mathrm{Spec}(B)$ be a local algebraization of a universal deformation of $E_{/\kappa}$. The data $(B^{\mathrm{h}}, \mathscr{E}_{/B^{\mathrm{h}}})$ is *uniquely* unique (as infinitesimal deformations of elliptic curves admit no non-trivial automorphisms, or in more fancy terms the moduli stack of elliptic curves is Deligne-Mumford). Does the action of the order 24 group $\Gamma = \mathrm{Aut}(E_{/\kappa})$ extend to an action of Γ on the pair $(B^{\mathrm{h}}, \mathscr{E}_{/B^{\mathrm{h}}})$? The main issue here is to determine when "twisting" a versal deformation by an automorphism at the residue field level does not destroy the versality property.

In order that one get satisfactory answers, we need to assume (in addition to the above running hypotheses, including formal faithfulness of F) that on the full subcategory $\mathscr{C}_S(\kappa)$ of artin local objects in $\widehat{\mathscr{C}}_S(\kappa)$, the automorphism functors of objects classified by F are formally unramified. That is, for an object A in $\mathscr{C}_S(\kappa)$ and an object ξ in $F(A)$ which induces ξ_0 in $F(\kappa)$, we suppose that the natural map of groups

$$\mathrm{Aut}_{F(A)}(\xi) \to \mathrm{Aut}_{F(\kappa)}(\xi_0) \tag{5.3}$$

is injective. This injectivity arises in the context of Deligne-Mumford stacks, so we make the following definition.

Definition 5.5. We say that F is *formally Deligne-Mumford at* ξ_0 when the map (5.3) is injective for every artinian deformation ξ of ξ_0 over $\mathscr{C}_S(\kappa)$.

Note that the injectivity of (5.3) is just a condition for artin local A in $\widehat{\mathscr{C}}_S(\kappa)$, and the conjunction of (5.1) being faithful and (5.3) being injective makes the F_{ξ_0} into a "set-valued" functor on the category local noetherian $\mathscr{O}_S$-algebras with residue field κ.

Remark 5.6. In Definition 5.5, we do not require $\mathrm{Aut}_{F(\kappa)}(\xi_0)$ to be finite.

Suppose now that F is formally Deligne-Mumford at ξ_0, so F_{ξ_0} is "set-valued". Let (B, ξ) be a local algebraization of a formal versal deformation $(C, (\xi_n))$ of ξ_0. If we pass to the structure $(B^{\mathrm{h}}, \xi^{\mathrm{h}})$ over the henselization of B, then the étale-local uniqueness of algebraizations (as in Theorem 5.3) and the fact that F_{ξ_0} is "set-valued" ensure that the henselized data $(B^{\mathrm{h}}, \xi^{\mathrm{h}})$ is unique up to *unique* isomorphism as a deformation of ξ_0. We will therefore refer to the pair $(B^{\mathrm{h}}, \xi^{\mathrm{h}})$ as the *henselized algebraization* of the formal versal deformation $(C, (\xi_n))$ of ξ_0. It is reasonable to now ask the following question:

- Does there exist an action of the abstract group
$$\Gamma_{\xi_0} \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{F(\kappa)}(\xi_0)$$
on the pair $(B^{\mathrm{h}}, \xi^{\mathrm{h}})$ lifting the action of Γ_{ξ_0} on ξ_0 (so in particular, Γ_{ξ_0} acts on the $\mathscr{O}_S$-algebra B^{h} in a manner which lifts the identity on its residue field κ)?

The fact that (5.1) is faithful and (5.3) is injective ensures the uniqueness of such an action if it exists. The answer to the existence part of the above general lifting question is affirmative in the *minimal* case:

Theorem 5.7. *Let S be an excellent scheme and let F, κ, and ξ_0 be as in Theorem* 1.5. *Assume that F is formally faithful and also that F is formally Deligne-Mumford at ξ_0. Let (B, ξ) be a henselized algebraization of a minimal formal versal deformation of ξ_0. Then there is a unique action of the group $\Gamma_{\xi_0} = \mathrm{Aut}_{F(\kappa)}(\xi_0)$ on (B, ξ) lifting its action on ξ_0.*

As an example, if ξ_0 is a polarized abelian variety or stable marked curve over a field, then its finite automorphism group canonically acts on any henselized algebraization of a universal formal deformation. Of course, this action is generally non-trivial on the henselian local base ring underlying such a henselized algebraization.

Proof. Due to the uniqueness (up to unique isomorphism) of henselized algebraizations of formal versal deformations of ξ_0, we claim that the problem is really one of making an action on minimal *formal* versal deformations. Namely, if $\iota : \xi \bmod \mathfrak{m}_B \simeq \xi_0$ is the implicit identification of ξ as a deformation of ξ_0 and σ is an $F(\kappa)$-automorphism of ξ_0, then we will use *minimality* to show that any map

$$\widehat{\varphi}_\sigma : \widehat{B} \to \widehat{B} \tag{5.4}$$

in $\widehat{\mathscr{C}}_S(\kappa)$ carrying the minimal formal versal deformation $(\widehat{B}, \widehat{\xi}, \iota)$ to the formal deformation $(\widehat{B}, \widehat{\xi}, \sigma \circ \iota)$ is an *isomorphism* of rings.

Grant for a moment that the $\widehat{\varphi}_\sigma$'s are isomorphisms. The isomorphism $\widehat{\varphi}_\sigma$ is *uniquely* determined, as our hypotheses on F force the triple $(\widehat{B}, \widehat{\xi}, \iota)$ to have no non-trivial automorphisms. If we didn't have the isomorphism condition on the $\widehat{\varphi}_\sigma$'s, then we would have no link with automorphisms and hence would not be able to establish uniqueness of the $\widehat{\varphi}_\sigma$'s (which is required to prove purely formally that $\widehat{\varphi}_\sigma$ and $\widehat{\varphi}_{\sigma^{-1}}$ are inverses to each other). Via the *isomorphism* $\widehat{\varphi}_\sigma$ we can view $(B, \xi, \sigma \circ \iota)$ as a henselized algebraization of the formal versal deformation $(\widehat{B}, \widehat{\xi}, \iota)$ corresponding to (B, ξ, ι). This could not be done without the crutch of having available an inverse to $\widehat{\varphi}_\sigma$.

By Theorem 5.3, we then conclude that there is an $\mathscr{O}_S$-algebra *isomorphism* φ_σ from (B, ξ, ι) to $(B, \xi, \sigma \circ \iota)$ which lifts the identity on κ and σ on $\xi_0 \in F(\kappa)$. The formal Deligne-Mumford property of F at ξ_0 implies that such an isomorphism φ_σ is *unique*, and the formal faithfulness of F ensures that φ_σ induces $\widehat{\varphi}_\sigma$. The isomorphisms φ_σ clearly provide the asserted unique action of Γ_{ξ_0}.

A natural idea for proving that $\widehat{\varphi}_\sigma$ is an isomorphism is to show that $\widehat{\varphi}_{\sigma^{-1}}$ is an inverse. However, since we do not know a priori that $(\widehat{B}, \widehat{\xi}, \sigma \circ \iota)$ is a *minimal* formal versal deformation (at least not before knowing that $\widehat{\varphi}_\sigma$ is an isomorphism), it appears difficult to show directly that the composites $\widehat{\varphi}_\sigma \circ \widehat{\varphi}_{\sigma^{-1}}$ and $\widehat{\varphi}_{\sigma^{-1}} \circ \widehat{\varphi}_\sigma$ are the identity on $\widehat{B}$. For this reason, it seems necessary to analyze the situation more carefully.

We have reduced ourselves to a general (and no doubt well-known) fact from deformation theory "with residue field extension". Here is the general setup. Let Λ be a complete local noetherian ring with residue field k (such as $\widehat{\mathscr{O}}_{S,s}$ above), κ a finite extension field of k, and $\widehat{\mathscr{C}}_\Lambda(\kappa)$ the category of complete local noetherian Λ-algebras with residue field κ. Let F be a category fibered in groupoids over the full subcategory $\mathscr{C}_\Lambda(\kappa)$ of artinian objects. Let ξ_0 be an object in $F(\kappa)$ and suppose that F satisfies the Schlessinger-Rim criteria at ξ_0 as in Definition 2.5, so a minimal formal versal deformation $(C, (\xi_n))$ of ξ_0 exists. Let (ξ'_n) in $\widehat{F}(C)$ be an object such that

- there is an abstract isomorphism $\iota_1 : \xi_1 \simeq \xi'_1$ in $F(C/\mathfrak{m}_C^2)$; we do *not* assume this to be a map in $F_{\xi_0}(C/\mathfrak{m}_C^2)$ (think of the case $\sigma \neq 1$ above).
- we are given some map

$$\widehat{\varphi} : C \to C \tag{5.5}$$

  in $\widehat{\mathscr{C}}_\Lambda(\kappa)$ carrying (ξ_n) over to (ξ'_n) but *not* necessarily respecting ι_1.

For example, $\widehat{\varphi}$ might be a map such that $\widehat{F}(\widehat{\varphi})$ respects some given structures of formal deformation of ξ_0, while $F(\iota_1)$ might not respect this same deformation structure. This is exactly the situation arising above with ι and $\sigma \circ \iota$, with (5.4) arising in the role of (5.5), so it suffices to show that (5.5) *must* be an isomorphism.

It suffices to check merely surjectivity of the local Λ-algebra map $\widehat{\varphi}$, and for this we immediately reduce to the case $\Lambda = k$ because the characterizing properties of minimal formal versal deformations make it evident that formation of such deformations respects replacing Λ by a non-zero quotient (such as k). We just have to check surjectivity of $\widehat{\varphi}$ as an endomorphism of $\overline{C} = C/\mathfrak{m}_C^2$. We can view $(\overline{C}, \overline{\xi}_1)$ as a "minimal" formal versal deformation of ξ_0 in the category $\mathscr{C}_{k,2}(\kappa)$ of finite local k-algebras with residue field κ and square zero maximal ideal. Thus, it is enough to consider the following claim.

Let B be an object in $\mathscr{C}_{k,2}(\kappa)$ and ξ an object in $F(B)$. If there exists a *surjection*

$$\pi : \overline{C} \to B \tag{5.6}$$

in $\mathscr{C}_{k,2}(\kappa)$ carrying $\overline{\xi}_1$ over to some ξ, then we claim that *any* morphism

$$\pi' : \overline{C} \to B \tag{5.7}$$

in $\mathscr{C}_{k,2}(\kappa)$ carrying $\overline{\xi}_1$ over to ξ must be a surjection. This is applied above by taking $B = \overline{C}$, $\xi = \xi_1'$, and π the identity endomorphism of $\overline{C}$ (and using the axiomatized isomorphism between ξ_1 and ξ_1' in $F(\overline{C})$ to make the identity map on $\overline{C}$ "carry" $\overline{\xi}_1$ over to ξ_1').

By using a κ-basis of the finite-dimensional κ-vector space $\mathfrak{m}_B$, we reduce the verification of the surjectivity of π' to the case where the artin local ring B has length 2. Thus, the only way that π' can *fail* to be surjective is if π' kills $\mathfrak{m}_{\overline{C}}$, in which case via π' the k-algebra B acquires a compatible structure of κ-*algebra* (respecting residue field identifications with κ), so $B \simeq \kappa[\varepsilon]$ in $\mathscr{C}_{k,2}(\kappa)$ and π' is exactly the composite of the canonical k-algebra maps

$$\overline{C} \twoheadrightarrow \kappa \hookrightarrow \kappa[\varepsilon].$$

Thus, ξ in the category $F(\kappa[\varepsilon])$ is isomorphic to $F(\pi')(\overline{\xi}_1)$, which in turn is isomorphic to the *trivial* deformation of the $F(\kappa)$-object

$$\overline{\xi}_1 \bmod \mathfrak{m}_{\overline{C}} \simeq \xi_0.$$

We conclude that the *surjection* (5.6) in $\mathscr{C}_{k,2}(\kappa)$ sends the object ξ_1 in $F_{\xi_0}(\overline{C})$ to the trivial point in the tangent space $t_{F_{\xi_0}} = \overline{F}_{\xi_0}(\kappa[\varepsilon])$. Note that the k-algebra surjection (5.6) to $B = \kappa[\varepsilon]$ is exactly determined by a *non-zero* κ-linear map

$$\Omega^1_{\overline{C}/k}/\mathfrak{m}_{\overline{C}} \to \kappa \cdot \varepsilon, \tag{5.8}$$

so we just have to show that such a non-zero map cannot induce a map (5.6) which sends ξ_1 to the trivial deformation in $t_{F_{\xi_0}}$. This is essentially the content of the minimality hypothesis on (ξ_n), but in order to check this in the presence of possible inseparability in the field extension κ/k it seems necessary to recall a couple of facts from the *construction* of minimal formal versal deformations in [SGA7, VI, 1.20] rather than to just argue purely in terms of (uni)versal properties.

Since minimal formal versal deformations are unique up to non-canonical isomorphism when they exist, and $(C, (\xi_n))$ only matters for our purposes up to non-canonical isomorphism, it is legitimate for us to replace this abstract pair with any specific construction of such a minimal deformation. Using the canonical isomorphism

$$\widehat{\Omega}^1_{C/\Lambda}/\mathfrak{m}_C \simeq \Omega^1_{\overline{C}/k}/\mathfrak{m}_{\overline{C}},$$

the *construction* of $(C, (\xi_n))$ in the proof of [SGA7, VI, 1.20] (which rests on the Schlessinger-Rim criteria at ξ_0 as in Definition 2.5) provides the existence of a certain finite-dimensional κ-vector space H (whose dual is a subspace of $t_{F_{\xi_0}}$ complementary to an "explicit" κ-subspace defined using $\Omega^1_{\kappa/k}$) and also the existence of a surjection

$$\overline{C} \twoheadrightarrow \kappa[H] \tag{5.9}$$

in $\mathscr{C}_{k,2}(\kappa)$ taking $\overline{\xi}_1$ to some θ_1 with the following two properties satisfied:

- the induced κ-linear map
$$\Omega^1_{\overline{C}/k}/\mathfrak{m}_{\overline{C}} \to \Omega^1_{\kappa[H]/k}/H \tag{5.10}$$
is an *isomorphism*;
- for any morphism $\kappa[H] \to \kappa[\varepsilon]$ in $\mathscr{C}_{k,2}(\kappa)$ which is *non-zero* on H (and *not* necessarily κ-linear), the induced map of κ-vector spaces
$$\overline{F}_{\xi_0}(\kappa[H]) \to \overline{F}_{\xi_0}(\kappa[\varepsilon]) = t_{F_{\xi_0}} \tag{5.11}$$
does *not* send the isomorphism class of θ_1 to the zero element (i.e., the isomorphism class of the trivial deformation of ξ_0).

Due to the isomorphism in (5.10) and the characterization of the surjective π in (5.6) in terms of a non-zero map (5.8), we conclude that π must factor through the surjection (5.9) inducing (5.10), so π induces a *surjective* map

$$\kappa[H] \twoheadrightarrow \kappa[\varepsilon] \tag{5.12}$$

in $\mathscr{C}_{k,2}(\kappa)$ such that the corresponding κ-vector space map

$$\overline{F}_{\xi_0}(\kappa[H]) \to \overline{F}_{\xi_0}(\kappa[\varepsilon]) = t_{F_{\xi_0}}$$

kills θ_1. Thus, by the general property described for (5.11), the surjective map (5.12) must *kill* H and hence (5.12) induces a k-algebra (but not necessarily κ-linear) *surjection* $\kappa \to \kappa[\varepsilon]$. A comparison of *finite* (non-zero) k-vector space dimensions of κ and $\kappa[\varepsilon]$ implies that no such surjection can exist. That is, (5.7) had to be surjective after all.

■

We conclude by addressing the natural question of whether we can bring a Γ-action on a henselization down to the level of a Γ-action on a residually trivial étale neighborhood. More specifically, we prove the following:

Theorem 5.8. *With notation and hypotheses as in Theorem* 5.7*, let* $(X, x; \xi)$ *be an algebraization of a minimal formal versal deformation of* ξ_0*. Assume also that the group* Γ_{ξ_0} *is finite. Then after passing to a residually trivial étale neighborhood of* $x \in X$*, there exists an action of* Γ_{ξ_0} *on the algebraization which lifts the action on* ξ_0*.*

Proof. We may assume $S = \mathrm{Spec}(R)$ is affine and $X = \mathrm{Spec}(B)$ is affine of finite type over S, with $x = \mathfrak{p} \in \mathrm{Spec}(B)$ a prime ideal, so (by Theorem 5.7) the base change of our algebraization to the henselized local ring $B_{\mathfrak{p}}^{\mathrm{h}}$ admits a Γ_{ξ_0}-action lifting that on ξ_0. We wish to "smear out" this action from the henselization down to some residually trivial étale neighborhood of $x \in X$.

Let $y_1, \dots, y_n \in B$ be R-algebra generators. For each $1 \le i \le n$, consider the conjugates $\gamma(y_i) \in B_{\mathfrak{p}}^{\mathrm{h}}$ for $\gamma \in \Gamma_{\xi_0}$. Define $s_{i,j}$ to be the jth symmetric polynomial in the $\gamma(y_i)$'s. The s_{ij}'s are finitely many elements in $B_{\mathfrak{p}}^{\mathrm{h}}$, and let A be the R-subalgebra of $B_{\mathfrak{p}}^{\mathrm{h}}$ generated by the s_{ij}'s. Clearly A is Γ_{ξ_0}-invariant. If we replace B with a sufficiently large étale neighborhood of $\mathfrak{p}$ contained in $B_{\mathfrak{p}}^{\mathrm{h}}$ which contains the finitely many s_{ij}'s, we lose the property of the old y_i's generating B, but we get ourselves to a situation where B is quasi-finite over an R-subalgebra A which is invariant under Γ_{ξ_0}. Renaming as $\mathfrak{p}$ the evident prime ideal of our modified B (coming from the maximal ideal of our unchanged henselized algebraization ring), let $\mathfrak{q}$ be the contraction of this prime to A.

There is a naturally induced map $A_{\mathfrak{q}}^{\mathrm{h}} \to B_{\mathfrak{p}}^{\mathrm{h}}$ which we claim is *finite*. Since $\mathrm{Spec}(B)$ is quasi-finite over $\mathrm{Spec}(A)$, it follows that $B \otimes_A A_{\mathfrak{q}}^{\mathrm{h}}$ is quasi-finite over the henselian local $A_{\mathfrak{q}}^{\mathrm{h}}$. By the structure theorem for quasi-finite separated schemes over a henselian local base [EGA, IV_4, 18.5.11], it follows that there is a unique local connected component of $\mathrm{Spec}(B \otimes_A A_{\mathfrak{q}}^{\mathrm{h}})$ which is finite over $\mathrm{Spec}(A_{\mathfrak{q}}^{\mathrm{h}})$ and contains the unique prime over $\mathfrak{p}$. This local component is visibly residually trivial and ind-local-étale over $\mathrm{Spec}(B_{\mathfrak{p}})$, yet it is also henselian (being finite over $\mathrm{Spec}(A_{\mathfrak{q}}^{\mathrm{h}})$). Hence, this component is uniquely (over $\mathrm{Spec}(B)$) isomorphic to $\mathrm{Spec}(B_{\mathfrak{p}}^{\mathrm{h}})$ as a scheme over $\mathrm{Spec}(A_{\mathfrak{q}}^{\mathrm{h}})$ (so indeed $B_{\mathfrak{p}}^{\mathrm{h}}$ is $A_{\mathfrak{q}}^{\mathrm{h}}$-finite).

With $B_{\mathfrak{p}}^{\mathrm{h}}$ now seen to be finite over a Γ_{ξ_0}-invariant R-subalgebra $A_{\mathfrak{q}}^{\mathrm{h}}$, we use standard direct limit arguments (working over residually trivial étale neighborhoods of $\mathfrak{q} \in \mathrm{Spec}(A)$) to make a base change by such a sufficiently large neighborhood to get to the case in which B is finite over a Γ_{ξ_0}-invariant finite type R-subalgebra A and

$$B_{\mathfrak{p}}^{\mathrm{h}} \simeq B \otimes_A A_{\mathfrak{q}}^{\mathrm{h}}.$$

Since F is locally of finite presentation, we can now run through standard direct limit arguments one more time to bring the Γ_{ξ_0}-action on the henselized algebraization $B_{\mathfrak{p}}^{\mathrm{h}}$ down to an action over $B \otimes_A A'$ for a sufficiently large residually finite étale neighborhood $\mathrm{Spec}(A')$ of $\mathfrak{q} \in \mathrm{Spec}(A)$. This gives what we wanted.

■

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
E-mail address: `bdconrad@umich.edu`

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail address: `dejong@math.mit.edu`