Math 249B. Geometric global class field theory

1. Introduction

Class field theory for global function fields K over finite constant fields k can be reformulated in purely algebro-geometric terms, as a theory of finite abelian coverings of smooth projective algebraic curves over finite fields (with controlled ramification over the base curve). That is, if X is the smooth projective and geometrically connected curve over k for which $k(X) \simeq K$ then one studies finite coverings $X' \to X$ of smooth projective curves whose corresponding function field extension K'/K is abelian. We have to interpret ramification and the notion of modulus of K in purely geometric terms.

Lang and Rosenlicht showed that this point of view, properly formulated with the aid of (not necessarily affine or projective) commutative algebraic groups, works over *any* perfect constant field k, not just finite fields, and in particular there is such a theory for curves X over fields k of characteristic 0, such as number fields (or $\mathbf{C}$)! In fact, the Lang-Rosenlicht theory with k a number field was used by Faltings in his proof of the Mordell conjecture. A caveat is required: there is an "ungeometric" feature (when k is not algebraically closed) corresponding to finite abelian coverings $X' \to X$ arising from nontrivial abelian extensions of the constant field (i.e., $k(X') = k' \otimes_k k(X)$ for a finite abelian extension k'/k, which is to say $X' = k' \otimes_k X$ as schemes). The Lang-Rosenlicht theory cannot say anything explicit about this, since for a general perfect constant field k it is hopeless to describe all possible k'/k (though for finite k it is not really a problem). In other words, the quotient $G_k^{\rm ab}$ of $G_K^{\rm ab}$ must somehow be "ruled out" from intervening in an explicit way in the theory with general perfect k. To do this, one has to somehow focus on coverings $X' \to X$ for which X' is *geometrically connected* over k, which is to say that k is algebraically closed in $k(X')$. Or rather, one needs to make sure that the theory keeps close track of geometric connectivity properties of various coverings.

This creates some mild complications, since it can happen for a pair of geometrically connected finite abelian coverings $X_1' \to X$ and $X_2' \to X$ over k that the composite abelian covering $X'' \to X$ (corresponding to the composite field $K_1'K_2'$ over K) has X'' *not* geometrically connected over k. In this handout we will not get into how this technical issue is handled in the proofs of the theory, and we refer the reader to Serre's book "Algebraic groups and class fields" for an exposition of the general theory. Unfortunately this book was written entirely in the archaic Weil style of algebraic geometry (making it very hard to read). Below we focus on giving an exposition of some main results in modern algebraic-geometric terms for geometrically connected abelian coverings, and we assume the reader has a good background in algebraic geometry (including the theory of Picard schemes). An excellent modern reference for the construction and structure of generalized Jacobians is Chapters 8 and 9 of the book "Néron models".

Remark 1.1. There is also a "geometric local class field theory", due to Serre, which describes abelian extensions of local fields with algebraically closed residue field (later generalized by Hazewinkel to any perfect residue field). He takes the viewpoint of studying the group of local units as a pro-algebraic group over the residue field, and replaces the role of the multiplicative group in the traditional local class field theory with a kind of fundamental group for this pro-algebraic group of units.

2. Generalized Jacobians

Let k be a perfect field and let X be a smooth projective geometrically connected curve over X. Places of $K = k(X)$ that are trivial on k are in natural one-to-one correspondence with closed points of X. We define a *modulus* $\mathfrak{m}$ on K exactly as for global fields, except that we only use places that are trivial on k (as is automatic when k is finite). Hence, a modulus is just another name for an effective divisor on X, and so it can be viewed as a finite closed subscheme of X (whose multiplicities at the points of its support are the corresponding exponents in the modulus).

Attached to any such $\mathfrak{m}$ with $\deg_k(\mathfrak{m}) > 1$ (i.e., $\mathfrak{m}$ as a divisor is non-empty and and is not a k-rational point) one can associate a projective and geometrically integral algebraic curve $X_{\mathfrak{m}}$ such that its normalization is $X \to X_{\mathfrak{m}}$ and it has a single non-smooth point that is moreover k-rational with pullback in X equal to $\mathfrak{m}$ (as a scheme). Roughly speaking, we form $X_{\mathfrak{m}}$ by scrunching $\mathfrak{m}$ into a single k-rational point, and the rigorous construction is done by a ring-theoretic fiber product operation (which we omit). The map $X \to X_{\mathfrak{m}}$ is universal for k-maps $X \to Y$ to arbitrary k-schemes such that $\mathfrak{m}$ scheme-theoretically factors through $Y(k)$. The simplest example is when $\mathfrak{m} = P + Q$ for a pair of distinct k-rational points $P, Q \in X$, in which case $X_{\mathfrak{m}}$ has a k-rational nodal singularity whose normalization has P and Q lying over the singularity. If $\mathfrak{m} = 2P$ for a k-point P of X then $X_{\mathfrak{m}}$ has a cuspidal singularity. A more subtle example is $\mathfrak{m} = P$ with $\deg_k(P) = 2$, in which case $X_{\mathfrak{m}}$ has a nodal singularity but its "tangent lines" at the singularity are not k-rational. (This is like non-split multiplicative reduction for an elliptic curve.)

By work of Grothendieck (but known in an earlier less precise form by Weil's contemporaries, such as Lang and Rosenlicht), for any geometrically integral projective scheme Y over k there is a naturally associated commutative k-group scheme $\mathrm{Pic}_{Y/k}$ locally of finite type over k that (roughly speaking) classifies line bundles on Y. Its formation commutes with any extension of the base field. This k-group is called the *Picard scheme* of Y, and if $\mathrm{char}(k) > 0$ it is generally non-reduced. However, when $\dim Y = 1$ it is always smooth (due to cohomological reasons). The identity component $\mathrm{Pic}^0_{Y/k}$ is quasi-projective and geometrically connected, and the so-called component group $\mathrm{Pic}_{Y/k}/\mathrm{Pic}^0_{Y/k}$ is an étale k-group scheme whose group of geometric points is finitely generated (Néron's Theorem of the Base). In case $\dim Y = 1$ the component group is the infinite cyclic group $\mathbf{Z}$ (via the "degree" of line bundles) and so we can naturally label the connected components as $\mathrm{Pic}^n_{Y/k}$ for $n \in \mathbf{Z}$; the nth one "classifies" degree-n line bundles on Y, and by the k-group structure it has a natural left action by $\mathrm{Pic}^0_{Y/k}$ that is simply transitive and free (in the scheme-theoretic sense). In other words, each $\mathrm{Pic}^n_{Y/k}$ is a principal homogeneous space for $\mathrm{Pic}^0_{Y/k}$ (and in particular is geometrically connected and quasi-projective over k). We will be interested in the case $n = 1$.

Example 2.1. If Y is a smooth curve then $\mathrm{Pic}^0_{Y/k}$ is the *Jacobian variety* of Y. This is an abelian variety whose dimension is the genus of Y. This has a very rich theory. When Y is not smooth then $\mathrm{Pic}^0_{Y/k}$ is generally not an abelian variety: roughly speaking, the singularities of Y contribute an "affine part". There are serious technical problems if k is not perfect (especially if the non-smooth locus has points that are not étale over k), so suppose now that k is perfect. In this case, if $\widetilde{Y} \to Y$ is the normalization (so it is smooth, by perfectness of k) then the induced "pullback" map $\mathrm{Pic}_{Y/k} \to \mathrm{Pic}_{\widetilde{Y}/k}$ identifies $\mathrm{Pic}^0_{\widetilde{Y}/k}$ as an abelian variety quotient of $\mathrm{Pic}^0_{Y/k}$ modulo a smooth connected affine k-group.

In particular, $\mathrm{Pic}^0_{Y/k}$ is affine if and only if Y has normalization of genus 0. For example, if Y is the nodal curve obtained from $\mathbf{P}^1_k$ by gluing 0 and ∞ to each other then $\mathrm{Pic}^0_{Y/k}$ is isomorphic to the affine k-group GL_1. The general structure of the affine part can be understood in terms of the structure of the singularities of Y (up to delicate issues related to the singular points perhaps not being k-rational; so really over $\overline{k}$ one can describe what is going on). For example, nodal singularities contribute a torus in the affine part, whereas more complicated singularities contribute unipotent pieces.

Now we focus our attention on the k-group varieties $J_{\mathfrak{m}} = \mathrm{Pic}^0_{X_{\mathfrak{m}}/k}$ for a varying modulus $\mathfrak{m}$ as above, with k now assumed to be *perfect*. The k-group $J_{\mathfrak{m}}$ is an extension of the Jacobian $J = \mathrm{Pic}^0_{X/k}$ by a smooth connected affine group $R_{\mathfrak{m}}$ which depends on $\mathfrak{m}$. For this reason, we call the $J_{\mathfrak{m}}$'s *generalized Jacobians* of X.

Example 2.2. In the special case $k = \mathbf{C}$ there is an analytic description:

$$J^{\mathrm{an}}_{\mathfrak{m}} \simeq \mathrm{H}^0(X, \Omega^1_{X/k}(\mathfrak{m}))^\vee / \mathrm{H}_1(X - \mathrm{supp}(\mathfrak{m}), \mathbf{Z});$$

in case $\mathfrak{m} = 0$ this recovers the classical analytic description of the Jacobian.

The focus of interest in the theory is to describe finite abelian coverings $X' \to X$ that are unramified away from $\mathfrak{m}$ and whose ramification structure is "no worse than $\mathfrak{m}$". In case $\mathfrak{m} = 0$ this is the problem of describing everywhere unramified finite abelian coverings $X' \to X$, and it was known classically how this should go, at least when X has positive genus (so there is an interesting theory) and X' is geometrically connected over k: there should be a unique étale isogeny of abelian varieties $J' \to J$ with constant kernel which pulls back to $X' \to X$ along a canonical map $X \to J$. (The constancy of the kernel corresponds to $J' \to J$ inducing a Galois extension of function fields over k, and not just over $\overline{k}$.) Actually, this is slightly incorrect when k is not algebraically closed, since if $X(k) = \emptyset$ then there is no canonical map $X \to J$ along which we can pull back the covering $J' \to J$ to get a covering of X. To get around this (since in practice we do not want to assume $X(k)$ is non-empty!), the right thing to work with is not a map $X \to J$ but rather a canonical map $X \to J^1 = \mathrm{Pic}^1_{X/k}$. (If $X(k)$ is non-empty we can use a k-rational point to translate J^1 into J within $\mathrm{Pic}_{X/k}$ so as to recover the more traditional viewpoint .) So the principal homogeneous spaces mentioned above are going to intervene due to the possibility that $X(k)$ may be empty.

The aim of the Lang-Rosenlicht theory is to do for general $\mathfrak{m}$ what the classical theory of the Jacobian does for $\mathfrak{m} = 0$: describe finite abelian (and geometrically connected!) coverings of X in terms of étale isogenies $G \to J_{\mathfrak{m}}$ (with constant kernel) of smooth connected commutative k-groups and compatible finite étale coverings $G^1 \to J^1_{\mathfrak{m}}$ of principal homogeneous spaces. In this way the "class field theory" of $K = k(X)$ is expressed in terms of the theory of étale isogenies of commutative algebraic groups (and principal homogeneous spaces for such groups), at least for the geometrically connected part. The theory of étale isogenies to $J_{\mathfrak{m}}$ can be studied by using the structure of $J_{\mathfrak{m}}$ (especially its abelian variety quotient J and its "affine part" $R_{\mathfrak{m}}$).

3. Main results

Consider X and $\mathfrak{m}$ as above. Let $J^n_{\mathfrak{m}} = \mathrm{Pic}^n_{X_{\mathfrak{m}}/k}$ for $n \in \mathbf{Z}$. There is a canonical map

$$\phi_{\mathfrak{m}} : X - \mathfrak{m} \to J^1_{\mathfrak{m}}$$

defined functorially (on S-valued points for any k-scheme S) by $x \mapsto \mathscr{O}(x)$ (where x is viewed as a section of $X_S := X \times_{\mathrm{Spec}\, k} S \to S$ supported away from $\mathfrak{m}_S$ and $\mathscr{O}(x)$ is the inverse ideal sheaf of this section). One can show that if $\deg_k(\mathfrak{m}) > 1$ then $\phi_{\mathfrak{m}}$ cannot be defined at any point of $\mathrm{supp}(\mathfrak{m})$, so as a rational map from X to $J^1_{\mathfrak{m}}$ its maximal domain of definition is in fact $X - \mathfrak{m}$.

The remarkable fact is that if $G \to J_{\mathfrak{m}}$ is an étale isogeny of smooth connected commutative k-groups and $G^1 \to J^1_{\mathfrak{m}}$ is a compatible map of principal homogeneous spaces (so it becomes an abelian covering space when $\ker(G \to J_{\mathfrak{m}})$ becomes constant, such as over $\overline{k}$) then in the pullback square

$$\begin{array}{ccc} U' & \longrightarrow & G^1 \\ \downarrow & & \downarrow \\ X - \mathfrak{m} & \xrightarrow[\phi_{\mathfrak{m}}]{} & J^1_{\mathfrak{m}} \end{array}$$

the left side is not only an abelian covering when viewed over $\overline{k}$ but U' is *geometrically connected* over k. This left side uniquely extends to a finite covering map $X' \to X$ of smooth projective geometrically connected curves, and it is étale away from $\mathrm{supp}(\mathfrak{m})$ so it is ramified at worst over the points in the support of $\mathfrak{m}$. In fact, the multiplicities in $\mathfrak{m}$ turn out to even "bound" how bad the ramification can be. This is all part of the following main results of the theory.

The first result is an analogue for the $\phi_{\mathfrak{m}}$'s of the Albanese property of Jacobians. It essentially classifies rational maps from curves to smooth connected commutative algebraic groups.

Theorem 3.1 (Rosenlicht)**.** *Let X be a smooth projective and geometrically connected curve over a perfect field k, and let $S \subseteq X$ be a finite set of closed points. Let H be a smooth connected commutative k-group, and let $f : X - S \to H^1$ be a map to a principal homogeneous space for H.*

There exists a modulus $\mathfrak{m}$ with $\mathrm{supp}(\mathfrak{m}) = X - \mathrm{dom}(f) \subseteq S$ *and a k-morphism $f^1_{\mathfrak{m}} : J^1_{\mathfrak{m}} \to H^1$ equivariant with respect to a k-group map $f_{\mathfrak{m}} : J_{\mathfrak{m}} \to H$ such that there is a commutative diagram*

$$\begin{array}{ccc} X - S & \xrightarrow{\phi_{\mathfrak{m}}} & J^1_{\mathfrak{m}} \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle f^1_{\mathfrak{m}}} \\ & & H^1 \end{array}$$

For any such $\mathfrak{m}$ the maps $f_{\mathfrak{m}}$ and $f^1_{\mathfrak{m}}$ are uniquely determined, and there is a unique such $\mathfrak{m} = \mathfrak{f}$ that divides all others. In particular, every f is "classified" by a triple $(\mathfrak{f}, f_{\mathfrak{f}}, H^1)$.

Moreover, any geometrically connected abelian covering of H^1 pulls back along f to a geometrically connected abelian covering of X that is unramified outside of $\mathrm{supp}(\mathfrak{m})$.

Theorem 3.2. *Let k be a perfect field and let X be a smooth projective and geometrically connected curve over k. Let $\pi : X' \to X$ be the finite covering map of smooth connected curves corresponding to a finite abelian extension K'/K of function fields (so X' may not be geometrically connected over k), and let $S \subseteq X$ be the ramification locus for this covering.*

There exists a modulus $\mathfrak{m}$ with support S and a connected finite abelian covering $G^1 \to J^1_{\mathfrak{m}}$ such that there is a cartesian diagram

$$\begin{array}{ccc} X' - \pi^{-1}(S) & \longrightarrow & G^1 \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ X - S & \xrightarrow[\phi_{\mathfrak{m}}]{} & J^1_{\mathfrak{m}} \end{array}$$

There is a unique such modulus $\mathfrak{f}_\pi$ which divides all others, in which case G^1 is geometrically connected if and only if X' is geometrically connected, and if k is finite then $\mathfrak{f}_\pi$ is the least admissible modulus $\mathfrak{f}_{K'/K}$ in the sense of class field theory for global function fields.

The formation of $\mathfrak{f}_\pi$ commutes with any extension of the ground field F/k, in the sense that we take a sum of moduli corresponding to the connected components of $X' \otimes_k F$ (each viewed as a connected finite abelian covering of $X \otimes_k F$).

In case the ground field k is perfect, the essential difficulty in the proof of class field theory – proving that the Artin map kills certain principal ideals – becomes easy to prove geometrically by means of the interpretation of geometric points of generalized Jacobians in terms of generalized ideal class groups. (More precisely, one has $J_{\mathfrak{m}}(k) = \mathrm{Cl}_{\mathfrak{m}}(K)$ when $\mathrm{Br}(k) = 1$, as happens when k is finite but not when k is a number field.)